Cyclic operads: syntactic, algebraic and categorified aspects

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Cyclic operads: syntactic, algebraic and categorified aspects

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Résumé

Opérades cycliques : aspects syntaxiques, algébriques et catégorifiés
par Jovana Obradović

Dans cette thèse, nous examinons différents cadres pour la théorie générale des opérades cycliques de Getzler et Kapranov. Comme le suggère le titre, nous établissons des fondements théoriques de natures syntaxiques, algébriques et catégorifiées pour la notion d'opérade cyclique.

Dans le traitement syntaxique, nous proposons un langage formel à la manière du λ-calcul, appelé μ-syntaxe, en tant que représentation légère de la structure ≪ entries-only ≫ d'opérades cycliques. Contrairement à la caractérisation originale des opérades cycliques, appelée la caractérisation ≪ exchangeable-output ≫, selon laquelle les opérations d'une opérade cyclique ont des entrées et une sortie qui peut être ≪ échangée ≫ avec une entrée, les opérades cycliques ≪ entries-only ≫ sont présentées comme des généralisations d'opérades pour lesquelles une opération n'a plus des entrées et une sortie, mais seulement des entrées (c'est-à-dire pour lesquelles la sortie est ≪au même niveau≫ que les entrées). Grâce aux méthodes de réécriture derrière le formalisme, nous donnons une preuve pas-à-pas complète de l'équivalence entre les définitions biaisées et non biaisées des opérades cycliques.

Guidés par le principe du microcosme de Baez et Dolan et par les définitions algébriques des opérades de Kelly et Fiore, dans l'approche algébrique, nous définissons les opérades cycliques à l'intérieur de la catégorie des espèces de structures de Joyal. De cette façon, la caractéristique originale ≪ exchangeable-output ≫ de Getzler et Kapranov, et la caractérisation alternative ≪ entries-only ≫ des opérades cycliques de Markl, sont toutes les deux incarnées comme ≪ monoïdes ≫ dans une catégorie ≪ monoïdale ≫ des espèces de structures. (À proprement parler, les deux produits sur les espèces, qui captent les deux façons de définir les opérades cycliques, ne sont pas monoïdaux, car ils ne sont pas associatifs, mais les structures induites apparaissent selon le même principe que celui qui reflète une spécification d'un monoïde dans une catégorie monoïdale. En particulier, ils sont tous les deux soumis à des isomorphismes qui compensent le défaut d'associativité.) En s'appuyant sur un résultat de Lamarche sur la descente pour les espèces, nous utilisons ces définitions ≪ monoïdales ≫ pour prouver l'équivalence entre les points de vue ≪ exchangeable-output ≫ et ≪ entries-only ≫ pour les opérades cycliques.

Enfin, nous établissons une notion d'opérade cyclique catégorifiée pour les opérades cycliques avec symétries, définies dans la catégorie des ensembles en termes de générateurs et relations. Les catégorifications que nous introduisons sont obtenues en remplaçant des ensembles d'opérations de la même arité par des catégories, en relâchant certains axiomes de la structure, comme l'associativité et la commutativité, en isomorphismes, tout en laissant l'équivariance stricte, et en formulant des conditions de cohérence pour ces isomorphismes. Le théorème de cohérence que nous prouvons a la forme ≪ tous les diagrammes d'isomorphismes canoniques commutent ≫. Pour les opérades cycliques ≪ entries-only ≫, notre preuve a un caractère syntaxique et s'appuie sur la cohérence des opérades non symétriques catégorifiées, établie par Došen et Petrić. Nous prouvons la cohérence des opérades cycliques ≪ exchangeable-output ≫, en ≪ relevant au cadre catégorifié ≫ l'équivalence entre les définitions ≪ entries-only ≫ et ≪ exchangeable-output ≫, mise en place précédemment dans l'approche algébrique.

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Abstract

Cyclic operads: syntactic, algebraic and categorified aspects

by Jovana Obradović

In this thesis, we examine different frameworks for the general theory of cyclic operads of Getzler and Kapranov. As suggested by the title, we set up theoretical grounds of syntactic, algebraic and categorified nature for the notion of a cyclic operad.

In the syntactic treatment, we propose a λ-calculus-style formal language, called μ-syntax, as a lightweight representation of the entries-only cyclic operad structure. As opposed to the original exchangeable-output characterisation of cyclic operads, according to which the operations of a cyclic operad have inputs and an output that can be "exchanged" with one of the inputs, the entries-only cyclic operads have only entries (i.e. the output is put on the same level as the inputs). By employing the rewriting methods behind the formalism, we give a complete step-by-step proof of the equivalence between the unbiased and biased definitions of cyclic operads.

Guided by the microcosm principle of Baez and Dolan and by the algebraic definitions of operads of Kelly and Fiore, in the algebraic approach we define cyclic operads internally to the category of Joyal's species of structures. In this way, both the original exchangeable-output characterisation of Getzler and Kapranov, and the alternative entries-only characterisation of cyclic operads of Markl are epitomised as "monoid-like" objects in "monoidal-like" categories of species. (Strictly speaking, the two products on species, which capture the two ways of defining cyclic operads, are not monoidal, as they are not associative, but the induced structures arise in the same way as the one reflecting a specification of a monoid in a monoidal category. In particular, they are both subject to isomorphisms which fix the lack of associativity.) Relying on a result of Lamarche on descent for species, we use these "monoid-like" definitions to prove the equivalence between the exchangeable-output and entries-only points of view on cyclic operads.

Finally, we establish a notion of categorified cyclic operad for set-based cyclic operads with symmetries, defined in terms of generators and relations. The categorifications we introduce are obtained by replacing sets of operations of the same arity with categories, by relaxing certain defining axioms, like associativity and commutativity, to isomorphisms, while leaving the equivariance strict, and by formulating coherence conditions for these isomorphisms. The coherence theorem that we prove has the form "all diagrams of canonical isomorphisms commute". For entries-only categorified cyclic operads, our proof is of syntactic nature and relies on the coherence of categorified operads established by Došen and Petrić. We prove the coherence of exchangeable-output categorified cyclic operads by "lifting to the categorified setting" the equivalence between entries-only and exchangeable-output cyclic operads, set up previously in the algebraic approach.

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Contents

To my family

Introduction

The notion of operad initially arose in the framework of algebraic topology, as a tool for characterising topological spaces that have the homotopy type of a k-fold loop space: it was shown by May in [May72] that these spaces are endowed with algebraic structure encoded by the *little k-disc operad*. Even though it was May who actualised the notion, thereby also creating the portmanteau *operad* by combining the words *operation* and *monad*, the same concept appeared earlier in the work [BV68] of Boardman and Vogt, under the name *operators in normal form*. The list of ancestors goes on: from PROPs and PACTs of Adams and Mac Lane [ML63], via *associahedron K* of Stasheff [Sta63] and *systems with compositions* of Lazard [Laz55], all the way back to the year 1898 and *complete algebraic systems* of Whitehead [Whi98].

The evolution that led to operads was all about studying composition of functions subject to some kind of associativity. In particular, it aimed at finding a language in which the related structures, which might be combinatorially cumbersome, can be presented in a way that clarifies their global configuration and allows to compare a priori different concepts. Here lies the importance of operads: they make a compact meta-algebraic setting convenient for materialising what is understood as the "type of algebras" and treating uniformly various algebraic problems. That an operad encodes a certain class of algebras means that its data consists of all operations with several arguments made of structure operations on such an algebra, which themselves form an analogous algebraic structure. In turn, the relations among operadic operations imply relations on the elements of an algebra. The way operads govern algebras in the same as the way theories govern their models.

The concept of an operad has been formalised in various ways, summarised in Table 1, whereby most of the characterisations admit themselves several flavours.

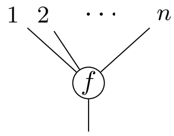
	Biased				
	Classical	Partial	Unbiased	Algebraic	Categorified
Operads	*Boardman, Vogt, May*	*Markl*	*Getzler, Jones*	*Kelly*	*Day, Street, Došen, Petrić, Dehling, Vallette*

Table 1: Various definitions of operads

The *biased* approach characterises an operad in terms of spaces of abstract operations of different arities, equipped with a notion of how to compose these operations and, if the operad is *symmetric*, an action of permuting their inputs. The composition structure is biased towards 2-*fold* operadic compositions, in the sense that the only explicitly defined concepts are "local" compositions of two operations. The various ways to derive k-*fold* operadic composition (i.e. a "global" composition of an assortment of k operations) are then equated by the appropriate associativiy axioms.

An n-ary operation f should be thought of as a single-node rooted tree, whose node is decorated with the symbol f and has n inputs, labeled either by natural numbers from 1 to n (in which case the operad is characterised as *skeletal*), or, equivalently, by elements of an arbitrary finite set of cardinality n (in which case the operad is *non-skeletal*).

Formally, in the skeletal approach, the space $\mathcal{O}(n)$ of n-ary operations of a symmetric operad is determined by a functor $\mathcal{O} : \mathbf{\Sigma}^{op} \to \mathbf{C}$, where $\mathbf{\Sigma}$ is the skeleton of the category $\mathbf{Bij}$ of finite sets and bijections, formed by the sets $[n] = \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, and $\mathbf{C}$ is an arbitrary symmetric monoidal category. Then, for any permutation σ of $[n]$, the induced map $\mathcal{O}(\sigma)$ determines a permutation of inputs of an operation



and this constitutes the action of the symmetric group $\mathbb{S}_n$ on $\mathcal{O}(n)$.

As for the formal description of operadic composition, the *classical* and *partial* characterisations provide two ways to complete the biased definition of an operad.

In the original definition of an operad, given by May in [May72], the operadic composition structure is specified by morphisms

$$\gamma_{n,k_1,\dots,k_n} : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n)$$

and the unit $id \in \mathcal{O}(1)$, defined for all $n \geq 1$ and $k_i \geq 0$, which are subject to associativity, equivariance and unit axioms. We refer to this kind of composition as *simultaneous*, since the morphisms $\gamma_{n,k_1,\dots,k_n}$ are to be thought of as simultaneous insertions of outputs of n operations into the n inputs of an n-ary operation.

The presence of operadic units allows for an equivalent biased approach for introducing operadic composition. Instead of working with simultaneous composition, one can introduce it by formulas

$$f \circ_i g = \gamma_{m,1,\dots,1,n,1,\dots,1}(f, id, \dots, id, g, id, \dots, id)$$

where g appears as the i-th argument of f, which specify individual compositions

$$\circ_i : \mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m + n - 1)$$

for all $1 \leq i \leq m$. This definition of operadic composition, which was first formalised by Markl in [Mar96], is called *partial*, since the morphisms $\circ_i$ are to be thought of as insertions of only one operation's output into the input i of another operation.

The non-skeletal variant of the symmetric operad structure is obtained by passing from $\mathbf{\Sigma}$ to $\mathbf{Bij}$, i.e. by building operadic composition over a functor $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{C}$.

Biased descriptions of an operad are the most explicit, but not the most compact ones.

In the *unbiased* (or *combinatorial*) framework, the operadic composition of an assortment of k operations, or, a k-fold operadic composition, is treated evenhandedly for all k, by defining an operad as an *algebra over a monad of rooted trees*. These trees act as pasting schemes for operations of an operad, and the operations decorating their nodes are "composed in one shot" through the structure morphism of the algebra. Operads were first described in these terms in [GJ94].

The *algebraic* (or *monoidal*) definition is another elegant characterisation of operads. The reference "algebraic" is there to indicate that such a definition most closely indicates the analogy between algebras and operads: in the algebraic framework, operads and algebras are both seen as monoids, but in a different monoidal category. While algebras are monoids in the monoidal category $(\mathbf{Set}, \otimes, \mathbb{K})$, operads are monoids in the monoidal category $(\mathbf{Set}^{\mathbf{Bij}^{op}}, \circ, I)$, whose monoidal product $\circ$, called the substitution product, arises as an instance of the Day convolution product [Day70]. In turn, the algebraic approach allows to extend various constructions inherent to algebraic homotopy theory, as are Koszul duality theory [GK94] and bar construction [GJ94], to the level of operads. The biased operad structure was assimiliated for the first time in the form of a monoid by Kelly in [Kel05] (the initial version of the paper dating from 1972). Independently, Joyal [Joy81] has studied the substitution product on the category of species of structures, which makes the context for the monoidal definition of $\mathbf{Set}$-based operads.

Last but certainly not least, for the purposes of higher-dimensional category theory and homotopy theory, categorification recently also emerged in operad theory, where at least three definitions of *categorified operads* have been proposed. In [DS01], Day and Street define *pseudo-operads* by categorifying the original "monoidal" definition of operads of Kelly [Kel05], which led to an algebraic, "one-line" characterisation of the form: *a pseudo-operad is a pseudo-monoid in a certain monoidal 2-category*. In [DP15], Došen and Petrić introduce the notion of *weak Cat-operad* by categorifying the biased definition of non-symmetric operads, which led them to an equational axiomatic definition, in the style of Mac Lane's definition of a monoidal category [ML98, Section XI.1]. Another analogy with their approach to categorification is given by bicategories of Bénabou [Bén67], in which the usual associativity and unit laws for composition of morphisms

$$(f \circ g) \circ h = f \circ (g \circ h), \qquad 1_A \circ f = f \qquad \text{and} \qquad f \circ 1_B = f$$

are replaced by the existence of coherent 2-isomorphisms

$$\beta : (f \circ g) \circ h \to f \circ (g \circ h), \qquad i_l : 1_A \circ f \to f \qquad \text{and} \qquad i_r : f \circ 1_B \to f.$$

In [DV15], Dehling and Vallette, through curved Koszul duality theory, obtain *higher homotopy symmetric operads*, for which the equivariance (ensuring that the symmetric structure agrees well with the composition structure) is also relaxed.

Each of these points of view on operads provides a particular approach to the treatment of algebras encoded by operads. The categories of algebras associated to operads are identified with categories of algebras whose operations have multiple inputs and one output (associative algebras, commutative algebras, Lie algebras).

The interest in encoding more general algebraic structures (for example, those acting on several underlying spaces, or those equipped with several algebraic operations, or those with multiple outputs), arising in areas as diverse as homological algebra, complex geometry, category theory and mathematical physics, led to the *renaissance of operads* in the early nineties of the last century. As an outcome, a wealth of examples and generalisations of operads came into existence. The latter include cyclic operads [GK95], modular operads [GK98], properads [Val04], dioperads [Gan03], etc. The earliest chronicle of the renaissance is given in [Lod94].

This thesis is about cyclic operads.

The formalism of cyclic operads was originally introduced by Getzler and Kapranov in [GK95], after ideas of Kontsevich [Kon94]. The motivation came from the framework of cyclic homology: in their paper, Getzler and Kapranov show that, in order to define cyclic homology for algebras encoded by an operad $\mathcal{O}$, $\mathcal{O}$ has to be what they call a cyclic operad. More precisely, the cyclic structure is exactly what is necessary for an operad $\mathcal{O}$ in order to obtain the framework in which invariant bilinear forms on algebras encoded by $\mathcal{O}$ can be handled. The cyclic homology of an algebra encoded by a cyclic operad then is defined as the non-abelian derived functor of the universal bilinear form.

Intuitively speaking, the endowment of the operad structure determined by the definition of a cyclic operad is provided by adding to the action of permuting the inputs of an operation an action of interchanging its output with one of the inputs. This feature essentially makes the distinction between the inputs and the output no longer visible, which is adequately captured by unrooted trees as pasting schemes for operations of a cyclic operad.

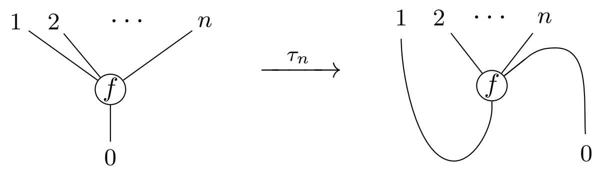
As it is the case for operads, the concept of cyclic operads comes in different flavours, as outlined by Table 2, which also gives a hint about the contents of this thesis.

The notion of a cyclic operad was originally given in the unbiased manner in [GK95, Definition 2.1], over the structure of a monad of unrooted trees. The advantage of the unbiased approach is reflected precisely in the transition from operads to cyclic operads: the definitions of various generalisations of operads can be obtained simply by shifting from rooted trees to other kinds of graphs.

	Biased					
	Classical	Partial		Unbiased	Algebraic	Categorified
		Exchangeable-output	Entries-only			
Cyclic Operads	*Getzler, Kapranov*	*Getzler, Kapranov, Markl*	*Markl* §2	*Getzler, Kapranov*	§3	§4

Table 2: Various definitions of cyclic operads

Like operads, biased cyclic operads can be defined by means of classical simultaneous composition [GK95, Theorem 2.2] or of partial composition [Mar08, Proposition 42]. In both of these definitions, the action of the symmetric group $\mathbb{S}_n$ (given by the symmetric operad structure) is extended with the cycle $\tau_n = (0, 1, \ldots, n)$, whose action includes making the output of an operation (denoted now with 0) to be the first input and the input indexed with n to be the output, in a way that is compatible with operadic composition and preserves units. The action of τ_n can be visualised as the clockwise rotation of all "wires" of a tree, such that each wire takes the position of its right-hand side neighboring wire:



The exchangeable-output feature of cyclic operads intuitively means that two operations can be composed along inputs that "used to be outputs" and outputs that "used to be inputs". This leads to another biased point of view on cyclic operads, in which they are seen as generalisations of operads for which an operation, instead of having inputs and an (exchangeable) output, now has only "entries", and it can be composed with another operation along any of them. Such an *entries-only definition* was first formalised by Markl in [Mar16, Definition 48], although the concept of partial compositions ${}_x\circ_y$, where x and y are the entries selected for composition, was already briefly introduced in [MSS02, Section 5.1]. By contrast, we refer to the definitions based on describing cyclic operads as operads with extra structure (accounting for the input-output interchange) as *exchangeable-output* ones.

What this thesis brings

This thesis follows the main line of the evolution of cyclic operads, summarises and relates different points of view for this notion, and most importantly, further furnishes the general theory of cyclic operads by introducing new means of describing them, from the syntactic (§2), algebraic (§3) and categorified (§4) standpoints, as indicated in the title and Table 2. It is outlined by the three main parts, each corresponding to a different foundational approach.

§2 Syntactic approach

In the spirit of recent years' movement in bringing closer mathematics and computer science communities through formalisation of mathematics, in the syntactic approach, we propose a λ-calculus-style formal language, called the μ-syntax, as a lightweight representation of the cyclic operad structure.

The name and the language of the μ-syntax formalism were motivated by another formal syntactical tool, the $\mu\tilde{\mu}$-subsystem of the $\overline{\lambda}\mu\tilde{\mu}$-calculus, presented by Curien and Herbelin in [CH00]. In their paper, programs are described by means of expressions called commands, of the form

$$\langle \mu\beta.c_1 \,|\, \tilde{\mu}x.c_2\rangle,$$

which exhibit a computation as the result of an interaction between a term $\mu\beta.c_1$ and an evaluation context $\tilde{\mu}x.c_2$, together with a symmetric reduction system

$$c_2[\mu\beta.c_1/x] \longleftarrow \langle \mu\beta.c_1 \,|\, \tilde{\mu}x.c_2\rangle \longrightarrow c_1[\tilde{\mu}x.c_2/\beta],$$

reflecting the duality between call-by-name reduction strategy (which evaluates the program first) and call-by-value reduction strategy (which evaluates the argument of the program first). In our syntactical approach, we follow this idea and view operadic composition as such a program, i.e. as an interaction between two operations f and g, where f, considered as a context for g, provides an input x (selected with $\tilde{\mu}$) for the output β of g (marked with μ). By moving this concept to the entries-only framework of cyclic operads, the input/output distinction of the $\mu\tilde{\mu}$-subsystem goes away, leading to the existence of a *single binding operator* μ, whose purpose is to select the entries of two operations which are to be connected in this interaction.

The advantage of the μ-syntax over the usual "mathematical" definitions of cyclic operads is tangible from two perspectives. On one hand, if one lays down the two usual ways of defining cyclic operads, the biased way ([GK95, Theorem 2.2], [Mar16, Definition 48], [Mar08, Proposition 42]), and the unbiased way ([GK95, Definition 2.1]), one would argue that these look quite formidable. This is due to the underlying intricate combinatorial structure of unrooted trees. The commands of the μ-syntax play the role of trees, but with the benefit of being rather simple in-line formulas. Accordingly, the equations of the μ-syntax make a crisp representation of the cumbersome laws defining the structure of cyclic operads. Summed up, the μ-syntax *makes the long story short(er)*.

On the other hand, in the spirit of Leibniz's *characteristica universalis* and *calculus ratiocinator*, the usefulness of the μ-syntax arises when the question about the completeness, rigour and formalisability of mathematical proofs is asked. This especially concerns long and involved proofs, which are common in operad theory. Such a proof is, for example, the proof of the equivalence between the biased and unbiased definitions of cyclic operads, which is a well-known result. The above requirements, typically asked for in computer science, reflect through this proof as follows. Firstly, in the operadic literature, incorporated in the structure of cyclic operads and similar definitions, one can find two formalisms of unrooted trees: in [GK95, Definition 2.1], the usual formalism of trees with "indivisible" edges is used, while in [Get09], [JK11], [KW17], trees with half-edges (or flags), due to [KM94], are used in the context of modular operads and Feynman categories. As there exist more than one tree formalism, there are also several proofs of the biased-unbiased equivalence (cf. [GK95, Theorem 2.2], [KW17, Section 5], [Man99, Section 4.2]). The *formalisability* property requires fixing a universal syntactic language in which the proof will be presented. The internal structural patterns of this language should be convenient for describing in a step-by-step fashion the transitions involved in this proof. In order to meet the *rigour* requirement, all the involved structures must be spelled out in detail. In particular, the correct treatment of the identities of the appropriate monad structure must be given, which is usually not the case in the literature. Finally, as required by the *completeness* property, the proof that the laws satisfied by an algebra over the monad indeed come down to the axioms from the biased definition, must be explicitly given. In order to illustrate how one gets closer to fulfilling these three requirements, we make a syntactic reformulation of the monad of unrooted trees figuring in the unbiased definition, which, together with the μ-syntax, makes a syntactic framework well-suited for a complete step-by-step proof of the equivalence.

§3 Algebraic approach

In our next approach, we exhibit cyclic operads within the algebraic framework of species of structures. A species of structures S associates to each finite set X a set $S(X)$ of combinatorial structures on X that are invariant under renaming the elements of X, in a way consistent with composition of such renamings. The notion, introduced in combinatorics by Joyal in [Joy81], has been set up to provide a description of discrete structures that is independent from any specific format these structures could be presented in. For example, $S(X)$ could be the set of graphs whose vertices are given by X, the set of all permutations of X, the set of all subsets of X, etc. Categorically speaking, a species of structures is simply a functor $S : \mathbf{Bij} \to \mathbf{Set}$. Species can be combined in various ways into new species and these "species algebras" provide the category of species with different notions of "tensor product". Some of these products, like the substitution product we mentioned earlier, allow to redefine operads internally to the category of species, as monoids.

The algebraic definition of symmetric operads of Kelly [Kel05] corresponds to the original biased definition of operads with simultaneous composition of May [May72]. This definition is, moreover, referred to as the *monoidal definition* of operads, since the substitution product constitutes a monoidal structure. The second algebraic definition, which characterises operads with partial composition, has been recently established by Fiore in [Fio14]. The *pre-Lie product* of Fiore's definition is not monoidal, but the inferred structure arises by the same kind of principle as the one reflecting a specification of a monoid in a monoidal category (which is why we call this definition the monoid-*like* definition of operads). This is an example of what has been called the *microcosm principle* by Baez and Dolan in [BD98]. The principle tells that

> *certain algebraic structures can be defined in any category equipped with a categorified version of the same structure,*

and the instance with monoids, presented in Table 3 below, can serve as a guide when seeking the most general way to internalize different algebraic structures.

	Monoidal category $\mathbf{M}$	Monoid $M \in \mathbf{M}$
product	$\otimes : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$	$\mu : M \otimes M \to M$
unit	$1 \in \mathbf{M}$	$\eta : 1 \to M$
associativity	$\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$	$\begin{tikzcd} (x\otimes x)\otimes x \arrow[rr, "\alpha_{x,x,x}"] \arrow[d, "\mu\otimes id"'] & & x\otimes(x\otimes x) \arrow[d, "id\otimes\mu"] \\ x\otimes x \arrow[dr, "\mu"'] & & x\otimes x \arrow[dl, "\mu"] \\ & x & \end{tikzcd}$
left unit	$\lambda_x : 1 \otimes x \to x$	$\begin{tikzcd} 1\otimes x \arrow[r, "\eta\otimes id"] \arrow[dr, "\lambda_x"'] & x\otimes x \arrow[d, "\mu"] & x\otimes 1 \arrow[l, "id\otimes\eta"'] \arrow[dl, "\rho_x"] \\ & x & \end{tikzcd}$
right unit	$\rho_x : x \otimes 1 \to x$	

Table 3: A monoid in a monoidal category

In our algebraic approach, we follow the microcosm principle in order to give two algebraic (and, more suggestively, monoid-like) definitions of cyclic operads, one for exchangeable-output cyclic operads, and the other for entries-only cyclic operads. This results in one-line conceptual descriptions, of the form

> *a cyclic operad is a monoid-like object in a certain monoidal-like category.*

Given that the monoidal-like categories in which cyclic operads "live" in these two definitions are built over the category of species, and, therefore, are of non-skeletal nature, we first propose a non-skeletal version of the biased definition of cyclic operads [Mar08, Proposition 42].

The main correspondence that we establish is the equivalence between these two algebraic definitions, which consolidates the equivalence between the entries-only and exchangeable-output points of view on cyclic operads.

§4 Categorified approach

In our categorified approach, we propose categorifications of entries-only and exchangeable-output cyclic operads with symmetries. With respect to the established notions of categorified operads, the style of our definitions corresponds to the one of [DP15], except that we also consider the action of the symmetric group. Yet, our categorified cyclic operads are not *cyclic operads up to the first level of homotopy* in the language of [DV15], as we keep equivariance strict.

Our process of categorification, like the one of [DP15], is the most common one: we replace sets (of operations of the same arity) with categories, obtaining in this way the intermediate notion of cyclic operad enriched over $\mathbf{Cat}$, followed by relaxing certain defining axioms of cyclic operads from equalities to isomorphisms, and exhibiting the conditions which make these isomorphisms coherent. In particular, the coherence theorem has the form "all diagrams made of canonical isomorphisms commute".

Concretely, for entries-only cyclic operads, the associativity and commutativity axioms

$$(f\,{}_{x}{\circ}_{\underline{x}}\,g)\,{}_{y}{\circ}_{\underline{y}}\,h = f\,{}_{x}{\circ}_{\underline{x}}\,(g\,{}_{y}{\circ}_{\underline{y}}\,h) \quad \text{and} \quad f\,{}_{x}{\circ}_{y}\,g = g\,{}_{y}{\circ}_{x}\,f$$

become the *associator* and *commutator* isomorphims, with instances

$$\beta^{x,\underline{x};y,\underline{y}}_{f,g,h} : (f\,{}_{x}{\circ}_{\underline{x}}\,g)\,{}_{y}{\circ}_{\underline{y}}\,h \to f\,{}_{x}{\circ}_{\underline{x}}\,(g\,{}_{y}{\circ}_{\underline{y}}\,h) \quad \text{and} \quad \gamma^{x,y}_{f,g} : f\,{}_{x}{\circ}_{y}\,g \to g\,{}_{y}{\circ}_{x}\,f,$$

respectively. At first glance, thanks to the (non-skeletal) equivariance axiom which "distributes" the action of the symmetric group from the composite of two operations to operations themselves, the coherence of the obtained notion seems easily reducible to the coherence of symmetric monoidal categories of Mac Lane (see [ML98, Section XI.1]): all diagrams made of instances of associator and commutator are required to commute. However, in the setting of cyclic operads, where the existence of operations is restricted, these instances do not exist for all possible indices, as opposed to the framework of symmetric monoidal categories. As a consequence, the coherence conditions that Mac Lane established for symmetric monoidal categories do not solve the coherence problem of categorified entries-only cyclic operads. In particular, the hexagon of Mac Lane is *not well-defined* in the setting of categorified entries-only cyclic operads. The coherence conditions that we do take from Mac Lane are the pentagon and the requirement that the commutator isomorphism is involutive. However, we need much more than this in order to ensure coherence. Borrowing the terminology from [DP15], we need two more *mixed* coherence conditions (i.e. coherence conditions that involve both associator and commutaor), a hexagon (which is *not* the hexagon of Mac Lane) and a decagon, as well as three more conditions which deal with the action of the symmetric group on *morphisms* of categories of operations of the same arity.

The approach we take to treat the coherence problem is of syntactic, term-rewriting spirit, as in [ML98] and [DP15], and relies on the coherence result of [DP15]. The proof of the coherence theorem consists of three faithful reductions, each restricting the coherence problem to a smaller class of diagrams, in order to finally reach diagrams that correspond exactly to diagrams of canonical isomorphisms of categorified non-symmetric skeletal operads, i.e. weak Cat-operads of [DP15]. Intuitively speaking, the first reduction excludes the action of the symmetric group, the second (and the most important) one removes "cyclicity", and the last one replaces non-skeletality with skeletality.

For exchangeable-output cyclic operads, the two associativity axioms of the underlying operad $\mathcal{O}$ become the *sequential associator* and *parallel associator* isomorphisms, with instances

$$\beta^{x,y}_{f,g,h} : (f \circ_x g) \circ_y h \to f \circ_x (g \circ_y h) \quad \text{and} \quad \theta^{x,y}_{f,g,h} : (f \circ_x g) \circ_y h \to (f \circ_y h) \circ_x g,$$

respectively. Therefore, the operadic part of the obtained structure is the non-skeletal and symmetric counterpart of a weak Cat-operad of [DP15]. However, in order to carry over the *equivalence between the entries-only and exchangeable-output cyclic operads*, set up previously in our algebraic approach, to the categorified setting (and, therefore, obtain, in the appropriate sense, the *correct* notion of categorified exchangeable-output cyclic operads), an axiom of the extra structure (accounting for the input-output exchange) must additionally be weakened. This leads to a third isomorphism, called the *exchange*, whose instances are

$$\delta^{z,x;v}_{f,g} : D_z(f \circ_x g) \to D_{zv}(g) \circ_v D_{xz}(f),$$

where $D_z(X) : \mathcal{O}(X) \to \mathcal{O}(X)$ is the endofunctor that "exchanges the input $z \in X$ with the output", and $D_{zy}(X) : \mathcal{O}(X) \to \mathcal{O}(X \backslash \{z\} \cup \{y\})$ is the functor that "exchanges the input $z \in X$ with the output and then renames it to y". We establish the coherence of this notion by "lifting" the proof of the equivalence (between the entries-only and exchangeable-output cyclic operads) of the algebraic treatment, thanks to the coherence of categorified entries-only cyclic operads.

The non-skeletal notion of exchangeable-output categorified cyclic operad described above can be straightforwardly coerced to a skeletal notion. In this way, a categorification of [Mar08, Proposition 42] is obtained. The coherence of the latter notion follows by "lifting" to the categorified setting the equivalence between non-skeletal and skeletal operads, established in [MSS02, Theorem 1.61], and extending it to the corresponding structures of categorified cyclic operads. In Appendix A.1, we provide details for the proof of [MSS02, Theorem 1.61].

Related to the coherence of skeletal exchangeable-output categorified cyclic operads, the skeletality requirement, combined with the presence of symmetries, causes an interesting issue, pointed to us by Petrić, which arises if one tries to give a coherence proof by means of rewriting. Namely, as opposed to non-skeletal equivariance, for skeletal equivariance *it is not possible* to "distribute" the action of the symmetric group from the composite of two operations to operations themselves. This makes the exclusion of symmetries (i.e. the first reduction mentioned earlier), at the very least, problematic. Therefore, as far as we can tell, the proof of skeletal coherence requires the transition to the non-skeletal framework (i.e. the equivalence of [MSS02, Theorem 1.61]), which shows that, when it comes to coherence with symmetries, the choice of non-skeletal framework is no longer a matter of convenience, but a matter of necessity. We illustrate this issue in Appendix A.2, where we also point out certain other merits of the non-skeletal operadic framework.

Chapter 1

Preliminaries

The purpose of this chapter is to recall the concepts from operad theory on which the rest of the thesis is built. This primarily means that we shall take a closer look at the axiomatics of biased definitions presented in Table 1 and Table 2. Nothing in this chapter is genuinely new: it serves merely to fix the vocabulary of these essential definitions, which slightly varies in the literature. Our main references are [GK95], [MSS02], [LV12] and [Mar16].

But before that, a few words on terminology are needed.

1.1 Notation and conventions

As we shall be mainly concerned with non-skeletal cyclic operads, we first focus on set-theoretical conventions. Then we give a glossary of key terms from type and rewriting theory.

1.1.1 About finite sets and bijections

Conforming to the computer science practice, in this thesis we assume that a sufficiently large universe of finite sets is fixed (countable is enough).

We shall use two different notions of union. In the category **Set** of sets and functions, for sets X and Y, $X+Y$ will denote the coproduct (disjoint union) of X and Y (constructed in the usual way by tagging X and Y, by, say, 1 and 2) and we shall use the notation $\Sigma_{i\in I}X_i$ (resp. $\Pi_{i\in I}X_i$) for the coproduct (resp. the Cartesian product) of the family of sets $\{X_i\,|\,i\in I\}$. In order to avoid making distinct copies of X and Y before taking the union, we take the usual convention of assuming that they are already disjoint. In the category **Bij** of finite sets and bijections, we shall denote the *ordinary* union of *already disjoint* sets X and Y with $X\cup Y$. In Appendix B, we compare these two ways of treating union, with a focus on what each of them brings in the context of (cyclic) operads.

If $f_1 : X_1 \to Z_1$ and $f_2 : X_2 \to Z_2$ are functions such that $X_1 \cap X_2 = \emptyset$ and $Z_1 \cap Z_2 = \emptyset$, $f_1 \cup f_2 : X_1 \cup X_2 \to Z_1 \cup Z_2$ will denote the function defined as f_1 on X_1 and as f_2 on X_2. If $Z_1 = Z_2 = Z$, we shall write $[f_1, f_2] : X_1 \cup X_2 \to Z$ for the function defined in the same way. Accordingly, for the corresponding functions between disjoint unions, we shall write $f_1 + f_2 : X_1 + X_2 \to Z_1 + Z_2$ and $[f_1, f_2] : X_1 + X_2 \to Z$.

For a bijection $\sigma : X' \to X$ and $Y \subseteq X$, we denote with $\sigma|^Y$ the restriction of σ on $\sigma^{-1}(Y)$.

For a bijection $\sigma : Y \to X$, we say that σ *renames* the variables of X to the appropriate variables of Y. In particular, if $\sigma : X\backslash\{x\}\cup\{y\} \to X$ is the identity on $X\backslash\{x\}$ and $\sigma(y) = x$, we say that σ renames x to y, and if $\tau : X \to X$ is identity on $X\backslash\{x,z\}$ and $\tau(x) = z$ and $\tau(z) = x$, we say that τ *exchanges* x and z.

We shall sometimes use the *cycle notation* for specifying permutations. For example, for $\sigma : \{x,y,z,u,v\} \to \{x,y,z,u,v\}$, defined by $\sigma(x) = y$, $\sigma(y) = x$, $\sigma(z) = z$, $\sigma(u) = v$ and $\sigma(v) = u$, the cycle notation presents σ as the product of its three disjoint cycles, i.e. as $\sigma = (x\ y)(u\ v)(z)$. We shall always omit the cycles of length 1 from this representation, i.e. we shall write $\sigma = (x\ y)(u\ v)$.

A decomposition of a finite set X is an ordered family $\{X_i\}_{i\in I}$ of (possibly empty) pairwise disjoint subsets of X such that their (ordinary) union is X. The attribute *ordered* is meant to indicate that the indexing set I is totally ordered, i.e. that, for example, the sets $\{\{x_1\},\{x_2,x_3\}\}$ and $\{\{x_2,x_3\},\{x_1\}\}$ constitute different decompositions of the set $\{x_1,x_2,x_3\}$.

1.1.2 Type-theoretical notions

Throughout the thesis, and notably during the syntactic and categorified treatement, we shall use basic notions and methods of type theory and rewriting theory. For a comprehensive account on these subjects, we refer to [Pie02] and [BN99]. We list here the essentials.

We assume given an infinite set $\mathtt{V}$ of *variables*, or *names* (countable is enough). We denote the variables of $\mathtt{V}$ by x, y, z, etc., possibly with indices and possibly underlined. We say that a variable x is *fresh with respect to a set* X if $x \notin X$. The existence of $\mathtt{V}$ ensures that, for any finite set, there exists a variable which is fresh with respect to that set.

A *multi-sorted formal theory* is a formal theory for which variables, constant symbols and function symbols, as well as all the terms built from them, have a property called *sort* or *type*. Types serve to control the formation of terms and to classify them. A model of a multi-sorted formal theory, i.e. of a typed formal language, is a model in the usual sense, which additionally takes into account sorts of the symbols of the signature of the theory. In other words, the domain of such a model is a collection of sets $\{\mathcal{M}(s_i)\}_{i\in I}$, indexed by all sorts of the theory, and the interpretation function maps constant symbols of sort s_i to the set $\mathcal{M}(s_i)$, for all $i \in I$, and function symbols of sort $(s_1,\ldots,s_n;s)$ to functions of the form $\mathcal{M}(s_1)\times\cdots\times\mathcal{M}(s_n)\to\mathcal{M}(s)$.

An *abstract rewriting system* (a rewriting system for short) is a pair $(A,\to)$, where A is a set and $\to$ is a binary relation on A. The name is supposed to indicate that an element (a,b) of $\to$ should be seen as a rewriting of a into b. We write $a\to b$ to denote that $(a,b)\in\to$. An element $a\in A$ is a *normal form* for $\to$ if there does not exist $a'\in A$, such that $a\to a'$. We say that a rewriting system $(A,\to)$ is *terminating* if there does not exist an infinite sequence $a_1\to a_2\to\cdots\to a_n\to\cdots$ of elements of A. We denote with $\xrightarrow{*}$ the reflexive and transitive closure of $\to$. A rewriting system $(A,\to)$ is *confluent* if, for any triple (a,a_1,a_2) of elements of A, such that $a\xrightarrow{*}a_1$ and $a\xrightarrow{*}a_2$, there exists $a'\in A$, such that $a_1\xrightarrow{*}a'$ and $a_2\xrightarrow{*}a'$. A rewriting system $(A,\to)$ is *locally confluent* if, for any triple (a,a_1,a_2) of elements of A, such that $a\to a_1$ and $a\to a_2$, there exists $a'\in A$, such that $a_1\xrightarrow{*}a'$ and $a_2\xrightarrow{*}a'$.

Fact 1. If $(A,\to)$ is terminating, then it is *normalising*, i.e. for any $a\in A$, there exists a normal form a', such that $a\xrightarrow{*}a'$.

Fact 2. If $(A,\to)$ is terminating and confluent, then for $a\in A$, there exists a *unique* normal form a', such that $a\xrightarrow{*}a'$.

Fact 3. If $(A,\to)$ is terminating, then it is confluent if and only if it is locally confluent.

1.2 Operads

The monographs [MSS02] and [LV12] both contain precise definitions of skeletal operads, in all variants presented in Table 1, except the categorified ones. Categorified skeletal operads in this thesis are the ones of [DP15].

We shall, however, mainly consider non-skeletal operads.

As our principal characterisation of operads we fix the *non-skeletal biased definition involving the symmetric group action and units, with operadic composition given in partial manner*. Moreover, we shall consider only **Set**-based operads. Whenever there is no risk of confusion, we shall refer to operads characterised in this way simply as operads. We shall speak of non-symmetric operads if we want to emphasise the absence of symmetries. Likewise, we shall speak of non-unital operads if we want to emphasise the absence of units.

We give now the precise definition, by transferring Markl's skeletal definition [Mar08, Proposition 42] to the non-skeletal framework. Below, for a functor $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Set}$, a bijection $\sigma : Y \to X$ and an element $f \in \mathcal{O}(X)$, we write f^{σ} for $\mathcal{O}(\sigma)(f)$.

Definition 1.1. An *operad* is a functor $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Set}$, together with a distinguished element $id_x \in \mathcal{O}(\{x\})$, called the *identity* or *unit* (indexed by x), that exists for each singleton $\{x\}$, and a *partial composition operation*

$$\circ_x : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X\backslash\{x\} \cup Y),$$

defined for arbitrary non-empty finite set X, an arbitrary finite set Y and an element $x \in X$, such that $X\backslash\{x\} \cap Y = \emptyset$. These data satisfy the axioms given below.

Associativity. For $f \in \mathcal{O}(X)$, $g \in \mathcal{O}(Y)$ and $h \in \mathcal{O}(Z)$, the following two equalities hold:

`[A1]` $(f \circ_x g) \circ_y h = f \circ_x (g \circ_y h)$, where $x \in X$ and $y \in Y$.

`[A2]` $(f \circ_x g) \circ_y h = (f \circ_y h) \circ_x g$, where $x, y \in X$, and

Equivariance. For bijections $\sigma_1 : X' \to X$ and $\sigma_2 : Y' \to Y$, and $f \in \mathcal{O}(X)$ and $g \in \mathcal{O}(Y)$, the following equality holds:

`[EQ]` $f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} g^{\sigma_2} = (f \circ_x g)^{\sigma}$, where $\sigma = \sigma_1|^{X\backslash\{x\}} \cup \sigma_2$.

Unitality. For $f \in \mathcal{O}(X)$ and $x \in X$, the following two equalities hold:

`[U1]` $id_y \circ_y f = f$, and

`[U2]` $f \circ_x id_x = f$.

Moreover, the unit elements are preserved under the action of $\mathcal{O}(\sigma)$, i.e.

`[UP]` ${id_x}^{\sigma} = id_u$, for any two singletons $\{x\}$ and $\{u\}$, and a bijection $\sigma : \{u\} \to \{x\}$.

For $f \in \mathcal{O}(X)$, we say that the elements of X are the *inputs* of f. An operad $\mathcal{O}$ is *constant-free* if $\mathcal{O}(\emptyset) = \emptyset$ □

For each of the axioms from Definition 1.1, we also (implicitly) assume the set disjointness conditions which ensure that all the partial compositions involved are a priori well-defined. We shall continue to omit mentioning explicitly these conditions whenever possible.

Remark 1.2. *Using* `[EQ]` *and* `[UP]`*, it can be easily shown that, for* $f \in \mathcal{O}(X)$ *and a renaming* $\sigma : X\backslash\{x\} \cup \{y\} \to X$ *of* x *to* y*, we have* $f \circ_x id_y = f^{\sigma}$.

Remark 1.3. *A* non-symmetric operad *is a collection of sets* $\mathcal{O}(X)$*, indexed by totally ordered finite sets* X*, equipped with a partial composition operation as in Definition 1.1, subject to the associativity and unitality axioms of Definition 1.1, all adapted naturally in a way which takes into account the ordering on indexing sets. See the proof of Theorem A.1 in Appendix A for details.*

A non-unital operad *structure is obtained from Definition 1.1 simply by forgetting the units and the unitality axioms.*

1.3 Cyclic operads

Given that cyclic operads originally emerged as operads with extra structure, they are typically presented in the skeletal exchangeable-output manner in the literature. Such a definition is [Mar08, Proposition 42]. The original unbiased definition [GK95, Definition 2.1] is of skeletal nature as well.

Our approach to cyclic operads will be the non-skeletal one, like for operads. In the literature, it is only the entries-only characterisation of cyclic operads that can be found in the non-skeletal presentation. The non-skeletal exchangeable-output definition of cyclic operads will be introduced for the first time in this thesis (in Chapter 3).

Therefore, the definition which links us with the literature, and the most important preliminary definition of this thesis, is the one of entries-only cyclic operads. Every approach to the treatment of cyclic operads in this thesis will have something to do with this definition. Whenever we say *entries-only cyclic operads*, it should be understood that we speak about *non-skeletal biased cyclic operads with the symmetric group action and units, whose composition structure is given in the entries-only manner and which are defined in* **Set**.

We give the precise definition below, by recalling Markl's definition [Mar16, Definition 48] for the particular case when the underlying functor is $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$, and adapting it further by also demanding units.

Definition 1.4. An *entries-only cyclic operad* is a functor $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$, together with a distinguished element $id_{x,y} \in \mathcal{C}(\{x,y\})$ for each two-element set $\{x,y\}$, called the *identity* or *unit* (indexed by $\{x,y\}$), and a *partial composition operation*

$${}_{x}\circ_{y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash\{x\} \cup Y\backslash\{y\}),$$

defined for arbitrary non-empty finite sets X and Y and elements $x \in X$ and $y \in Y$, such that $X\backslash\{x\} \cap Y\backslash\{y\} = \emptyset$. These data satisfy the axioms given below.

Sequential associativity. For $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$, $h \in \mathcal{C}(Z)$, $x \in X$, $\underline{x}, y \in Y$ and $\underline{y} \in Z$, the following equality holds:

(A1) $(f\,{}_{x}\circ_{\underline{x}}\, g)\,{}_{y}\circ_{\underline{y}}\, h = f\,{}_{x}\circ_{\underline{x}}\, (g\,{}_{y}\circ_{\underline{y}}\, h)$.

Commutativity. For $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$, $x \in X$ and $y \in Y$, the following equality holds:

(CO) $f\,{}_{x}\circ_{y}\, g = g\,{}_{y}\circ_{x}\, f$.

Equivariance. For bijections $\sigma_1 : X' \to X$, $\sigma_2 : Y' \to Y$ and $\sigma = \sigma_1|^{X\backslash\{x\}} \cup \sigma_2|^{Y\backslash\{y\}}$, and $f \in \mathcal{C}(X)$ and $g \in \mathcal{C}(Y)$, the following equality holds:

(EQ) $f^{\sigma_1}\,{}_{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)}\, g^{\sigma_2} = (f\,{}_{x}\circ_{y}\, g)^{\sigma}$.

Left unitality. For $f \in \mathcal{C}(X)$, $x \in X$ and a bijection σ that renames x to z, the following equality holds:

(U1) $id_{y,z}\,{}_{y}\circ_{x}\, f = f^{\sigma}$.

Moreover, the unit elements are preserved under the action of $\mathcal{C}(\sigma)$, i.e.

(UP) ${id_{x,y}}^{\sigma} = id_{u,v}$,

for any two two-element sets $\{x,y\}$ and $\{u,v\}$, and a bijection $\sigma : \{u,v\} \to \{x,y\}$.

For $f \in \mathcal{C}(X)$, we say that the elements of X are the *entries* of f. An entries-only cyclic operad $\mathcal{C}$ is *constant-free* if $\mathcal{C}(\emptyset) = \mathcal{C}(\{x\}) = \emptyset$, for all singletons $\{x\}$. □

Note that we impose a slightly weaker condition on the sets X and Y and elements $x \in X$ and $y \in Y$ involved in partial composition than in [Mar16, Definition 48]: instead of requiring X and Y to be disjoint, as Markl does, we allow the possibility that they intersect, provided that their intersection is a subset of $\{x,y\}$. This also means that we allow the possibility that $x = y$. Nevertheless, the characterizations of Definition 1.4 and [Mar16, Definition 48] (with units added), are equivalent. More precisely, all partial compositions allowed by [Mar16, Definition 48] are obviously covered by the Definition 1.4. As for the other direction, if $f\,{}_{x}\circ_{y}\, g$ is such that, say, $x \in X \cap (Y\backslash\{y\})$, then we can define $f\,{}_{x}\circ_{y}\, g$ as $f^{\sigma}\,{}_{x'}\circ_{y}\, g$, where x' is chosen outside of Y, and $\sigma : (X\backslash\{x\}) \cup \{x'\} \to X$ is identity everywhere except on x', which is sent to x, obtaining in this way a valid definition in the sense of [Mar16, Definition 48]. As for the units, here is a notational remark.

Notation 1.5. *It is understood that* $id_{x,y} = id_{y,x}$. *We reserve the notation* $id_{\{x,y\}}$ *for the identity bijection on the two-element set* $\{x, y\}$.

The lemma below gives basic properties of the partial composition operation.

Lemma 1.6. *The partial composition operation from Definition 1.4 satisfies the following laws.*

Parallel associativity. *For* $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$, $h \in \mathcal{C}(Z)$, $x, y \in X$, $\underline{x} \in Y$ *and* $\underline{y} \in Z$, *the following equality holds:*

`(A2)` $(f \; {}_{x}{\circ}_{\underline{x}} \; g) \; {}_{y}{\circ}_{\underline{y}} \, h = (f \; {}_{y}{\circ}_{\underline{y}} \; h) \; {}_{x}{\circ}_{\underline{x}} \, g.$

Right Unitality. *For* $f \in \mathcal{C}(X)$, $x \in X$ *and a bijection* σ *that renames* x *to* z, *the following two equalities hold:*

`(U2)` $f \; {}_{x}{\circ}_{y} \; id_{y,z} = f^{\sigma}.$

Proof. In order to prove `(A2)`, we combine `(CO)` and `(A1)`, as follows:

$$(f \; {}_{x}{\circ}_{\underline{x}} \; g) \; {}_{y}{\circ}_{\underline{y}} \, h = (g \; {}_{\underline{x}}{\circ}_{x} \; f) \; {}_{y}{\circ}_{\underline{y}} \, h = g \; {}_{\underline{x}}{\circ}_{x} \, (f \; {}_{y}{\circ}_{\underline{y}} \, h) = (f \; {}_{y}{\circ}_{\underline{y}} \; h) \; {}_{x}{\circ}_{\underline{x}} \; g.$$

For `(U2)`, we combine `(CO)` and `(U1)` in the obvious way. ■

It is easy to show that the axiom `(EQ)` can be expressed in the way presented in the following lemma.

Lemma 1.7. *Let* $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$, $x \in X$ *and* $y \in Y$ *be such that* $X\backslash\{x\} \cap Y\backslash\{y\} = \emptyset$, *and let* $\sigma : Z \to X\backslash\{x\} \cup Y\backslash\{y\}$ *be an arbitrary bijection. If* $\tau_1 : X' \to X, \tau_2 : Y' \to Y$ *and* $\tau : Z \to X'\backslash\{\tau_1^{-1}(x)\} \cup Y'\backslash\{\tau_2^{-1}(y)\}$ *are bijections, such that* $\sigma = (\tau_1|^{X\backslash\{x\}} \cup \tau_2|^{Y\backslash\{y\}}) \circ \tau$, *then the following equality holds:*

$$(f \, {}_{x}{\circ}_{y} \; g)^{\sigma} = (f^{\tau_1} \, {}_{x}{\circ}_{y} \; g^{\tau_2})^{\tau}.$$

In the following two remarks, we point out to an axiomatisation for partial composition operations ${}_{x}{\circ}_{y}$ alternative to the one of Definition 1.4 and we discuss the axiomatisations of cyclic operads without units.

Remark 1.8. *Related to the implication* `(A1)`$\wedge$`(CO)`$\Rightarrow$`(A2)` *proved in Lemma 1.6, it is also true (and easily checked) that* `(A2)`$\wedge$`(CO)`$\Rightarrow$`(A1)`. *Moreover, we also have that* `(A2)`$\wedge$`(U1)`$\Rightarrow$`(CO)`*:*

$$f \; {}_{x}{\circ}_{y} \; g = (id_{x,y} \; {}_{y}{\circ}_{x} \; f) \, {}_{x}{\circ}_{y} \; g = (id_{x,y} \; {}_{x}{\circ}_{y} \; g) \, {}_{y}{\circ}_{x} \; f = g \, {}_{y}{\circ}_{x} \; f \, .$$

These observations show that the equalities

`(A2)`, `(EQ)`, `(U1)` *and* `(UP)`

provide an equivalent and smaller *axiomatisation for* ${}_{x}{\circ}_{y}$. *It is precisely this axiomatisation that we shall use in Chapter 3, for the algebraic approach to cyclic operads. For the syntactic treatment of Chapter 2, will shall use the axiomatisation of Definition 1.4.*

Remark 1.9. *Although the axiomatisations of the partial composition operation* ${}_{x}{\circ}_{y}$ *from Definition 1.4 and Remark 1.8 are equivalent, when it comes to non-unital cyclic operads, it is not the case that* both of them *are adapted simply by forgetting the structure of units (and omitting the unitality axioms). This is true only for Definition 1.4, whose non-unital modification, therefore, contains the following axioms:*

`(A1)`, `(CO)` *and* `(EQ)`.

As for the axiomatisation from Remark 1.8, if one excludes the unitality axioms `(U1)` *and* `(UP)`*, the commutativity* `(CO)` must be put back as a primitive axiom*, in order to be able to derive the sequential associativity* `(A1)`*, indispensable for the structure of a cyclic operad. Therefore, in this case, the non-unital axiomatisation is given by*

`(A2)`, `(CO)` *and* `(EQ)`.

The two non-unital axiomatisations are clearly equivalent. As our primary definition of non-unital entries-only cyclic operads, whose categorification we introduce in Chapter 4, we fix the one obtained by removing units from Definition 1.4.

Although we do not encounter non-symmetric cyclic operads in this thesis, in the remark below we point out to their definition as well. This will be usefull at some point in Chapter 4, where a *nonexample* arises by depriving cyclic operads completely from the action of the symmetric group.

Remark 1.10. *In view of (and in contrast to) Remark 1.3, one should have in mind that non-symmetric cyclic operads still contain* cyclic actions *and the appropriate equivariance axiom. For their precise definition, we refer to [CGR14, Section 3.2] (exchangeable-output, skeletal) and [Mar16, Sections 1,2,3] (entries-only, non-skeletal).*

Finally, a morphism of (cyclic) operads is a natural transformation between the underlying functors, which preserves the structure of partial compositions and units. Cyclic operads and morphisms between them form a category $\mathbf{CO_{en}}$ (the "en" indicating their entries-only nature).

Chapter 2

A formal language for cyclic operads

Now that the entries-only definition of cyclic operads is settled, the intuition about the *μ-syntax*, which we present in this chapter, can be made more tangible. Referring to Definition 1.4 and the foresight made in the Introduction, the pattern $\langle \mu x.\_ \,|\, \mu y.\_ \rangle$, inherent to the μ-syntax, is crafted in order to encode the partial composition operation $(-)_x{\circ}_y(-)$. Thus, from the tree-wise perspective, $\langle \mu x.\_ \,|\, \mu y.\_ \rangle$ describes the procedure of constructing an unrooted tree by grafting two unrooted trees along entries (or half-edges, or flags) x and y. From a more general combinatorial point of view, this construction (and, in particular, the syntactic concept of binding) can also be understood in terms of differentiation of species, as a mapping $\partial S \cdot \partial S \rightarrow S$, where ∂S is the derivative of the species S and $\cdot$ denotes the product of species. In fact, it is precisely this mapping that we shall use in Chapter 3, in order to define cyclic operads internally to the category of species. In addition to commands of the form $\langle \mu x.\_ \,|\, \mu y.\_ \rangle$, which describe *partial* grafting of two unrooted trees, the μ-syntax features another kind of commands, whose shape is $(-)\{\mu x.\_, \dots, \mu y.\_\}$, and which describe *simultaneous* grafting of unrooted trees. Such a command encodes the procedure of constructing an unrooted tree by grafting to *all the entries* of the corolla $(-)$ the unrooted trees within the brackets, along their respective entries bound by μ. Therefore, the command $(-)\{\mu x.\_, \dots, \mu y.\_\}$ is to the command $\langle \mu x.\_ \,|\, \mu y.\_ \rangle$ what the original notion of simultaneous operadic composition of [May72] is to the notion of partial operadic composition of [MSS02], but in the framework of cyclic operads. Finally, the equations of the μ-syntax identify different constructions on unrooted trees that should be regarded the same.

This chapter develops as follows. In Section 2.1, we extract, from the structure of partial compositions of Definition 1.4, the notion of simultaneous composition. In Section 2.2, we recast cyclic operads of Definition 1.4 as models of the equational theory whose syntax of terms describes the free cyclic operad generated by a collection of operations. Note that this still does not involve the μ-syntax. This straightforward reformulation puts Definition 1.4 in the syntactic context in which it can get formally related to the μ-syntax. In Section 2.3, we move to this context the unbiased definition [GK95, Definition 2.1]. In particular, this involves a syntactic reformulation and a detailed description of the monad of unrooted trees. This section finishes with the theorem that expresses the equivalence between biased and unbiased cyclic operads. Section 2.4 will be devoted to the introduction and analysis of the μ-syntax. We conclude this section by exhibiting the correspondence between the μ-syntax formalism and the unbiased characterisation of cyclic operads. In Section 2.5, we employ the μ-syntax in crafting the proof of the equivalence stated in Section 2.3.

In the remainder of the chapter, merely for the sake of simplicity, we restrict ourselves to constant-free cyclic operads, to which we shall refer simply as cyclic operads.

2.1 Simultaneous entries-only composition

A cyclic operad $\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$, specified by Definition 1.4, naturally incorporates the concept of *simultaneous composition*, as a sequence of partial compositions of the form as in the law (A2) from Lemma 1.6, that is, in which the entry involved in the next instance of a composition always comes from a fixed operation $f \in \mathcal{C}(X)$ and which, moreover, ends when *all* the entries of $f \in \mathcal{C}(X)$ are exhausted. In order to avoid writing explicitly such sequences, we introduce the following notation. For $f \in \mathcal{C}(X)$, let

$$\varphi : x \mapsto (Y_x, g_x, \underline{x})$$

be an assignment that associates to each $x \in X$ a finite set Y_x, an operation $g_x \in \mathcal{C}(Y_x)$ and an element $\underline{x} \in Y_x$, in such a way that

$$\bigcap_{x \in X} Y_x \backslash \{\underline{x}\} = \emptyset.$$

Let, moreover, $\sigma : X' \to X$ be an arbitrary bijection such that for all $x \in X$,

$$X' \backslash \{\sigma^{-1}(x)\} \cap Y_x \backslash \{\underline{x}\} = \emptyset.$$

Under these assumptions, the composite assignment

$$\varphi \circ \sigma : x' \mapsto (Y_{\sigma(x')}, g_{\sigma(x')}, \underline{\sigma(x')}),$$

defined for all $x' \in X'$, together with $f^\sigma \in \mathcal{C}(X')$, determines the composition

$$((f^\sigma \; {}_{x'}\circ_{\underline{\sigma(x')}} \, g_x) \; {}_{y'}\circ_{\underline{\sigma(y')}} \, g_y) \; {}_{z'}\circ_{\underline{\sigma(z')}} \, g_z \cdots ,$$

consisting of a sequence of partial compositions indexed by the entries of f^σ. We will use the abbreviation $f^\sigma(\varphi \circ \sigma)$ to denote such a composition. Thanks to (A2), $f^\sigma(\varphi \circ \sigma)$ does not depend on the order in which the partial compositions were carried out. We finally set

$$f(\varphi) = f^\sigma(\varphi \circ \sigma), \tag{2.1.1}$$

and refer to $f(\varphi)$ as *the simultaneous composition determined by f and φ*. That $f(\varphi)$ does not depend on the choice of σ is a consequence of (EQ).

Notice that without the renaming role of σ, $f(\varphi)$ is not necessarily well-defined. For example, $f(\varphi) = (f \; {}_x\circ_{\underline{x}} \, g_x) \; {}_y\circ_{\underline{y}} \, g_y$, where $f \in \mathcal{C}(\{x, y\})$, $g_x \in \mathcal{C}(\{\underline{x}, y\})$ and $g_y \in \mathcal{C}(\{\underline{y}, v\})$, is not well-defined, although φ satisfies the required disjointness condition.

In relation to the above construction, the statements of the following lemma are easy consequences of (EQ).

Lemma 2.1. *The simultaneous composition $f(\varphi)$ has the following properties.*

1. *Let $\psi : Z \to \bigcup_{x \in X}(Y_x \backslash \{\underline{x}\})$ be a bijection such that for all $x \in X$, $\underline{x} \notin \psi^{-1}(Y_x \backslash \{\underline{x}\})$. Denote with $\psi_{\underline{x}}$ the extension on Y_x of the bijection $\psi|^{Y_x \backslash \{\underline{x}\}}$, which is identity on $\underline{x}$, and let φ_ψ be defined as $\varphi_\psi : x \mapsto (g_x^{\psi_{\underline{x}}}, \underline{x})$, for all $x \in X$. Then $f(\varphi)^\psi = f(\varphi_\psi)$.*

2. *Let $\psi : y \mapsto (h_y, \underline{y})$ be an assignment that associates to each $y \in \bigcup_{x \in X}(Y_x \backslash \{\underline{x}\})$ an operation $h_y \in \mathcal{C}(Z_y)$ and $\underline{y} \in Z_y$, in such a way that $f(\varphi)(\psi)$ is defined. If φ_ψ is the assigment defined as $\varphi_\psi : x \mapsto (g_x^{\psi_{\underline{x}}}, \underline{x})$, where $\psi_{\underline{x}}$ denotes the extension on Y_x of the assignment $\psi|_{Y_x \backslash \{\underline{x}\}}$, which is identity on $\underline{x}$, then $f(\varphi)(\psi) = f(\varphi_\psi)$.*

2.2 Biased definition of cyclic operads: combinator syntax

The generators-and-relations nature of Definition 1.4 allows us to easily formalise cyclic operads as models of the multi-sorted equational theory which we now introduce.

The signature of this theory is determined by taking as sorts all finite sets (of cardinality at least 2, when modelling constant-free cyclic operads), while, having denoted with s the sort of a variable or a constant symbol and with $(s_1, \ldots, s_n; s)$ the sort of an n-ary function symbol, as constant symbols we take the collection consisting of

$$id_{x,y} : \{x, y\}$$

and, as function symbols, we take the collection consisting of

$$\sigma : (Y; X) \text{ (of arity 1)} \quad \text{and} \quad {}_x\circ_y : (X, Y; X \backslash \{x\} \cup Y \backslash \{y\}) \text{ (of arity 2)},$$

where $x, y \in \mathtt{V}$ and σ ranges over all bijections of finite sets. Here, $\mathtt{V}$ is the infinite set of variables whose existence we postulated in the Introduction.

Fixing a collection of *sorted variables*, or *parameters* P, and denoting with $P(X)$ the collection of parameters whose sort is X, the terms of the theory are built in the usual way:

$$\boxed{s, t ::= a \ | \ id_{x,y} \ | \ s \,{}_x\circ_y\, t \ | \ t^{\sigma}}$$

whereas the assignment of sorts to terms is done by the rules given in Figure 2.1.

$$\frac{a \in P(X)}{a : X} \qquad \frac{x \neq y}{id_{x,y} : \{x, y\}} \qquad \frac{t : X \quad \sigma : (Y; X)}{t^{\sigma} : Y}$$

$$\frac{s : X \quad t : Y \quad x \in X \quad y \in Y \quad X \backslash \{x\} \cap Y \backslash \{y\} = \emptyset}{s \,{}_x\circ_y\, t : X \backslash \{x\} \cup Y \backslash \{y\}}$$

FIGURE 2.1: Typing rules for the terms of the equational theory modeled by cyclic operads

The equations of the theory are derived from the axioms of Definition 1.4, and there are two additional equations, namely

$$id_{x,y}^{\sigma} = id_{u,v} \qquad \text{and} \qquad (t^{\sigma})^{\tau} = t^{\sigma \circ \tau}, \tag{2.2.1}$$

where, in the first equation, $\sigma : (\{u, v\}; \{x, y\})$.

We can now reformulate the entries-only definition of cyclic operads as follows:

A cyclic operad is a model of the equational theory from above.

That this characterisation indeed describes the same structure as does Definition 1.4 is clear from the requirements that models of multi-sorted theories fulfill. The domain of such a model is a collection of sets $\mathcal{C}(X)$, arising by interpreting all sorts X, and the interpretation of the remaining of the signature in this universe exhibits the cyclic operad structure in the obvious way. Observe that the equations (2.2.1) ensure that the assignment $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$, induced by the model, is functorial.

Let $\underline{\mathcal{C}} : \mathbf{Bij}^{op} \to \mathbf{Set}$ be a functor and let

$$P_{\underline{\mathcal{C}}} = \{a \in \underline{\mathcal{C}}(X) \,|\, X \text{ is a finite set}\} \tag{2.2.2}$$

be the collection of *parameters of* $\underline{\mathcal{C}}$. Observe that $P_{\underline{\mathcal{C}}}$ can be considered as a collection of sorted variables for the equational theory introduced above. In this regard, we call the syntax of terms built over $P_{\underline{\mathcal{C}}}$ the *combinator syntax generated by* $\underline{\mathcal{C}}$ and we refer to terms as *combinators*. We shall denote the set of all combinators induced by $\underline{\mathcal{C}}$ by $\mathtt{cTerm}_{\underline{\mathcal{C}}}$, and, for a finite set X, $\mathtt{cTerm}_{\underline{\mathcal{C}}}(X)$ will be used to denote the set of all combinators of type X.

We finish this part with a notational remark. If $\mathcal{C}$ is a cyclic operad (and, hence, a model of the equational theory from above), and if $\underline{\mathcal{C}}$ is the underlying functor of $\mathcal{C}$, we shall denote with $[\_]_{\mathcal{C}} : \mathtt{cTerm}_{\underline{\mathcal{C}}} \to \mathcal{C}$ the induced interpretation of the combinator syntax.

2.3 Unbiased definition: a syntax for the monad of unrooted trees

In this part, we syntactically reformulate the unbiased definition [GK95, Definition 2.1]. The adaptations we make also include translating it to the non-skeletal setting, and reconstructing it within a formalism of unrooted trees that incorporates edges as pairs of half-edges, due to [KM94]. As it will be clear in Section 2.4, the formal language of unrooted trees that we present here is crafted in a way which reflects closely the formal language of the μ-syntax.

2.3.1 Graphs and unrooted trees

Let $\underline{\mathcal{C}} : \mathbf{Bij}^{op} \to \mathbf{Set}$ be a functor such that $\underline{\mathcal{C}}(\emptyset) = \underline{\mathcal{C}}(\{x\}) = \emptyset$, for all singletons $\{x\}$, and let $P_{\underline{\mathcal{C}}}$ be as in (2.2.2). *The syntax of unrooted trees generated by* $P_{\underline{\mathcal{C}}}$ is synthesised as follows.

An *ordinary corolla* is a term

$$a(x_1, \ldots, x_n),$$

where $a \in \underline{\mathcal{C}}(X)$ and $X = \{x_1, \ldots, x_n\}$. We refer to a as the *head symbol* of $a(x_1, \ldots, x_n)$. We call the elements of X the *free variables* of $a(x_1, \ldots, x_n)$, and we write $FV(a) = X$ to denote this set. Whenever the set of free variables is irrelevant, we shall refer to an ordinary corolla only by its head symbol.

In addition to ordinary corollas, we define *special corollas* to be terms of the shape

$$(x, y),$$

i.e. terms which do not have a parameter as a head symbol and which consist only of two distinct variables $x, y \in \mathtt{V}$. For a special corolla (x, y), we define $FV((x, y)) = \{x, y\}$.

Remark 2.2. *In both ordinary and special corollas, the order of appearance of free variables in the terms is irrelevant. In other words, we consider equal the terms, say, $a(x, y, z)$ and $a(z, x, y)$, as well as (x, y) and (y, x).*

Definition 2.3. A *graph* $\mathcal{V}$ is a finite set of (ordinary and special) corollas with mutually disjoint free variables, together with an involution σ on the set

$$V(\mathcal{V}) = \bigcup_{i=1}^{k} FV(a_i) \cup \bigcup_{j=1}^{p} FV((u_j, v_j))$$

of all variables occuring in $\mathcal{V}$. We write

$$\mathcal{V} = \{a_1(x_1, \ldots, x_n), \ldots, a_k(y_1, \ldots y_m), \ldots, (u_1, v_1), \ldots, (u_p, v_p); \sigma\}.$$

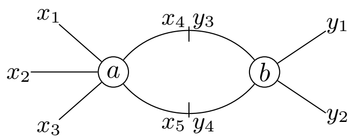
We denote with $Cor(\mathcal{V})$ the set of all corollas of $\mathcal{V}$. The set of *edges* $Edge(\mathcal{V})$ of $\mathcal{V}$ consists of pairs of variables x and y, such that $x \neq y$ and $\sigma(x) = y$ (and, therefore, also $\sigma(y) = x$). We denote the egdes by (xy). Naturally, it is understood that (xy) and (yx) are the same edges. Finally, we refer to the fixpoints of σ as the *free variables of* $\mathcal{V}$, the set of which we shall denote with $FV(\mathcal{V})$. $\square$

Remark 2.4. *The set of variables of a graph in our formalism corresponds to the set of* flags *(or half-edges) in the formalism of [KM94]. Such a formalism is inherent to operad theory. In graph theory in general, one does not usually encounter graphs with half-edges: graphs typically feature "indivisible" edges. The* *graphs considered here can be viewed as graphs with an* interface, *provided by the half-edges which are not paired to form edges (see [FDC13]).*

Remark 2.5. *The condition $\underline{\mathcal{C}}(\emptyset) = \underline{\mathcal{C}}(\{x\}) = \emptyset$, imposed on $\underline{\mathcal{C}} : \mathbf{Bij}^{op} \to \mathbf{Set}$, corresponds to the convention to consider only constant-free cyclic operads. For the general case, the syntax of graphs is straightforwardly supplemented with appropriate corollas: ordinary corollas will additionally contain terms of the form $a(x)$, corresponding to elements $a \in \mathcal{C}(\{x\})$, and there will be another kind of corollas, corresponding to elements $a \in \underline{\mathcal{C}}(\emptyset)$, denoted simply with a. If a graph contains only corollas of the latter kind, the involution of the graph is set to be the empty function.*

Here is an example.

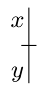
Example 2.6. The graph $\{a(x_1, x_2, x_3, x_4, x_5), b(y_1, y_2, y_3, y_4\}; \sigma\}$, where $\sigma = (x_4\ y_3)(x_5\ y_4)$, should be depicted as



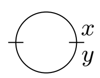
This graph has two corollas, $a(x_1, x_2, x_3, x_4, x_5)$ and $b(y_1, y_2, y_3, y_4)$, two edges, (x_4, y_3) and (x_5, y_4), and five free variables, x_1, x_2, x_3, y_1, y_2. ☐

And here is another example, this time with *more exotic* graphs, in the sense that they do not contain ordinary corollas.

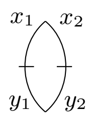
Example 2.7. The simplest example of a graph without ordinary corollas is a graph with a single special corolla and the identity involution, say $\{(x, y); id_{\{x,y\}}\}$. We depict it as



By changing the identity involution to $\sigma(x) = y$, we obtain an *ordinary-corolla-free loop*:



Finally, the graph $\{(x_1, y_1), (x_2, y_2), \sigma\}$, where $\sigma(x_1) = x_2$ and $\sigma(y_1) = y_2$ is the simplest example of an *ordinary-corolla-free wheel* (i.e. an ordinary-corolla-free *cycle*):



☐

Graphs do not need to be connected. *Connected graphs* are distinguished by the following recursive definition:

- ◇ for any finite set X, $a \in \underline{\mathcal{C}}(X)$ and involution σ on X, $\{a(x_1, \dots, x_n); \sigma\}$ is connected,
- ◇ for any two-element set $\{x, y\}$ and involution σ on $\{x, y\}$, $\{(x, y); \sigma\}$ is connected,
- ◇ if graphs $\mathcal{V}_1$ and $\mathcal{V}_2$, with involutions σ_1 and σ_2, respectively, are connected, and if $V(\mathcal{V}_1) \cap V(\mathcal{V}_2) = \emptyset$, then, for any $x \in FV(\mathcal{V}_1)$ and $y \in FV(\mathcal{V}_2)$, the graph $\mathcal{V}$, determined by $Cor(\mathcal{V}) = Cor(\mathcal{V}_1) \cup Cor(\mathcal{V}_2)$ and the involution σ on $V(\mathcal{V})$, defined by

$$\sigma(v) = \begin{cases} \sigma_1(v), & \text{if } v \in V(\mathcal{V}_1) \backslash \{x\} \\ \sigma_2(v), & \text{if } v \in V(\mathcal{V}_2) \backslash \{y\} \\ y, & \text{if } v = x \end{cases}$$

  is connected.

Remark 2.8. *Returning to Remark 2.5, if one allows corollas of the form a, where $a \in \underline{\mathcal{C}}(\emptyset)$, then a graph containing such a corolla is connected only if it does not contain any other corollas.*

For a graph $\mathcal{V}$ with involution σ and corollas $c, d \in Cor(\mathcal{V})$, a *path from c to d* is a finite sequence of variables $(v_1, \ldots, v_{2n})$, $n \geq 1$, such that $v_i \in V(\mathcal{V})$, for all $1 \leq i \leq 2n$, $v_1 \in FV(c)$, $v_{2n} \in FV(d)$, and there exist corollas $c_1, \ldots, c_{2n-2} \in Cor(\mathcal{V})$, such that $v_{j+1}, v_{j+2} \in FV(c_i)$, for all $1 \leq j \leq 2n-2$, and $\sigma(v_{2k-1}) = v_{2k}$, for all $1 \leq k \leq n$ (for $n = 1$, there are trivially zero such corollas). The *length* of the path $(v_1, \ldots, v_{2n})$ is $2n$ and v_1 and v_2 are its *ending variables*.

The *set of subgraphs of a graph $\mathcal{V}$ (with involution σ)* is obtained by the following recursive definiton:

- if $a(x_1, \ldots, x_n) \in Cor(\mathcal{V})$, then $\{a(x_1, \ldots, x_n); id_X\}$, where $X = \{x_1, \ldots, x_n\}$, is a subgraph of $\mathcal{V}$,
- if $(x, y) \in Cor(\mathcal{V})$, then $\{(x, y); id_{\{x,y\}}\}$ is a subgraph of $\mathcal{V}$,
- if graphs $\mathcal{V}_1$ and $\mathcal{V}_2$, with involutions σ_1 and σ_2, respectively, are subgraphs of $\mathcal{V}$, and if

$$E(\mathcal{V}_1, \mathcal{V}_2) = \{(xy) \,|\, x \in FV(\mathcal{V}_1), y \in FV(\mathcal{V}_2) \text{ and } \sigma(x) = y\}$$

  is the set of all edges of $\mathcal{V}$ "between $\mathcal{V}_1$ and $\mathcal{V}_2$", then, for any subset $E \subseteq E(\mathcal{V}_1, \mathcal{V}_2)$, $E = \{(x_i y_i) \,|\, i \in I\}$, the graph $\mathcal{W}_E$, determined by $Cor(\mathcal{W}_E) = Cor(\mathcal{V}_1) \cup Cor(\mathcal{V}_2)$ and the involution τ_E on $V(\mathcal{W}_E)$, defined by

$$\tau_E(v) = \begin{cases} \sigma_1(v), & \text{if } v \in V(\mathcal{V}_1) \backslash \{x_i \,|\, i \in I\} \\ \sigma_2(v), & \text{if } v \in V(\mathcal{V}_2) \backslash \{y_i \,|\, i \in I\} \\ \sigma(v), & \text{otherwise} \end{cases}$$

  if $Cor(\mathcal{V}_1) \cap Cor(\mathcal{V}_2) = \emptyset$, and by

$$\tau(v) = \begin{cases} \sigma_1(v), & \text{if } v \in V(\mathcal{V}_1) \backslash (\{x\} \cup \{x_i \,|\, i \in I\}) \\ \sigma_2(v), & \text{if } v \in V(\mathcal{V}_2) \backslash (\{y\} \cup \{y_i \,|\, i \in I\} \bigcup_{a \in Cor(\mathcal{V}_2) \cap Cor(\mathcal{V}_1)} FV(a)) \\ \sigma(v), & \text{otherwise} \end{cases}$$

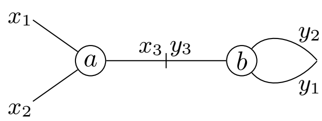
  if $Cor(\mathcal{V}_1) \cap Cor(\mathcal{V}_2) \neq \emptyset$, where $x \in FV(\mathcal{V}_1)$ and $y \in V(\mathcal{V}_2) \backslash FV(\mathcal{V}_2)$ are (the unique variables) such that $\sigma(x) = y$, is a subgraph of $\mathcal{V}$.

Observe that, just as graphs do not have to be connected, so do not subgraphs of an arbitrary graph.

Starting from this notion of graph, an *extended unrooted tree* is defined as a *connected graph without loops, multiple edges and cycles*. As the latter three requirements are standard in the terminology of graphs, we omit their formal definition and illustrate them with examples instead.

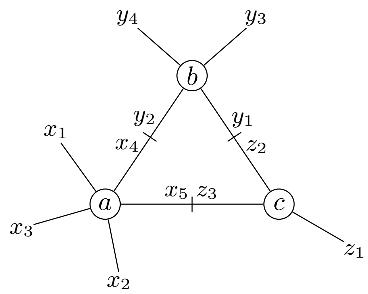
EXAMPLE 2.9. The graph from EXAMPLE 2.6 is not an extended unrooted tree, since it has two edges between corollas a and b.

The graph $\{a(x_1, x_2, x_3), b(y_1, y_2, y_3); \sigma\}$, where $\sigma = (x_3\ y_3)(y_1\ y_2)$, is not an extended unrooted tree either, since the edge (y_1, y_2) connects the corolla b with itself, i.e. it is a loop:



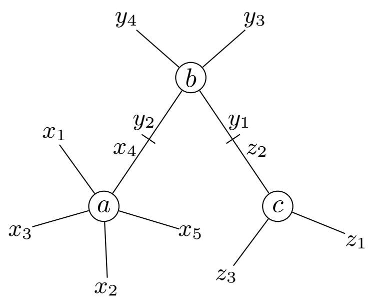
Exceptionally, but for the same reason, the graph $\{(x, y); \sigma\}$, where $\sigma(x) = y$, from Example 2.7, is not an extended unrooted tree.

The graph $\{a(x_1, x_2, x_3, x_4, x_5), b(y_1, y_2, y_3, y_4), c(z_1, z_2, z_3); \sigma\}$, with $\sigma = (x_4\ y_2)(y_1\ z_2)(z_3\ x_5)$, is another example of a graph which is not an extended unrooted tree, this time because of the presence of a cycle that connects its three corollas:



In particular, for the same reason, the graph $\{(x_1, y_1), (x_2, y_2); \sigma\}$, where $\sigma = (x_1\ x_2)(y_1\ y_2)$, from Example 2.7, is not an extended unrooted tree. □

Example 2.10. We get an example of a graph which is an extended unrooted tree by changing the involution σ of the graph $\{a(x_1, x_2, x_3, x_4, x_5), b(y_1, y_2, y_3, y_4), c(z_1, z_2, z_3); \sigma\}$ from Example 2.9, to, say, $\sigma' = (x_4\ y_2)(y_1\ z_2)$, producing in this way the extended unrooted tree with graphical representation



□

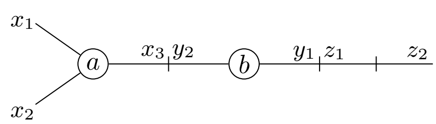
In moving from graphs to trees, we shall additionally differentiate the classes of extended unrooted trees with respect to the shape of corollas they contain. Let $\mathcal{T}$ be a connected graph with no loops, multiple edges and cycles.

- If $Cor(\mathcal{T})$ consists only of ordinary corollas, then $\mathcal{T}$ is an *ordinary unrooted tree*.
- If $Cor(\mathcal{T})$ is a singleton with a special corolla, then $\mathcal{T}$ is an *exceptional unrooted tree*.
- An *unrooted tree* is either an ordinary unrooted tree or an exceptional unrooted tree.

Example 2.11. The graph from Example 2.10 is an ordinary unrooted tree.

The graph $\{(x, y); id_{\{x,y\}}\}$ (see also Example 2.7) is an exceptional unrooted tree.

The graph $\{a(x_1, x_2, x_3), b(y_1, y_2), (z_1, z_2); \sigma\}$, where $\sigma = (x_3\, y_2)(y_1\, z_1)$, depicted as



is an extended unrooted tree. It is neither ordinary, nor exceptional. □

Remark 2.12. *Observe that, for an ordinary unrooted tree $\mathcal{T}$ and any two corollas $a, b \in Cor(\mathcal{T})$, there exists a unique path from a to b.*

A *subtree* of an (extended) unrooted tree $\mathcal{T}$ is a connected, non-empty subgraph of $\mathcal{T}$. We say that a subtree $\mathcal{S}$ of $\mathcal{T}$ is *proper* if $Cor(\mathcal{S}) \neq Cor(\mathcal{T})$. We say that two subtrees $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{T}$ are *adjacent*, and we write $a_{\mathcal{T}}(\mathcal{S}_1, \mathcal{S}_2) = 1$, if there exist $u \in FV(\mathcal{S}_1)$ and $v \in FV(\mathcal{S}_2)$, such that $\sigma(u) = v$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are not adjacent, we write $a_{\mathcal{T}}(\mathcal{S}_1, \mathcal{S}_2) = 0$.

A *decomposition of an (extended) unrooted tree* $\mathcal{T}$ (*with involution* σ) is a set of subtrees of $\mathcal{T}$ defined recursively as follows:

- ◇ $\{\mathcal{T}\}$ is a decomposition of $\mathcal{T}$,
- ◇ if $\mathcal{T}_1$ and $\mathcal{T}_2$ are subtrees of $\mathcal{T}$ with involutions σ_1 and σ_2, respectively, such that $Cor(\mathcal{T}_1) \cap Cor(\mathcal{T}_2) = \emptyset$, $Cor(\mathcal{T}) = Cor(\mathcal{T}_1) \cup Cor(\mathcal{T}_2)$ and there exist $x \in FV(\mathcal{T}_1)$ and $y \in FV(\mathcal{T}_2)$ such that
$$\sigma(v) = \begin{cases} \sigma_1(v), & \text{if } v \in V(\mathcal{T}_1)\backslash\{x\} \\ \sigma_2(v), & \text{if } v \in V(\mathcal{T}_2)\backslash\{y\} \\ y, & \text{if } v = x, \end{cases}$$
and if $\{\mathcal{T}_{11}, \ldots, \mathcal{T}_{1n}\}$ is a decomposition of $\mathcal{T}_1$ and $\{\mathcal{T}_{21}, \ldots, \mathcal{T}_{2m}\}$ is a decomposition of $\mathcal{T}_2$, then $\{\mathcal{T}_{11}, \ldots, \mathcal{T}_{1n}, \mathcal{T}_{21}, \ldots, \mathcal{T}_{2m}\}$ is a decomposition of $\mathcal{T}$. In particular, we shall write
$$\mathcal{T} = \{\mathcal{T}_1\,(xy)\,\mathcal{T}_2\} \tag{2.3.1}$$
to indicate that $\{\mathcal{T}_1, \mathcal{T}_2\}$ is a decomposition of $\mathcal{T}$ ("along" the edge (xy)).

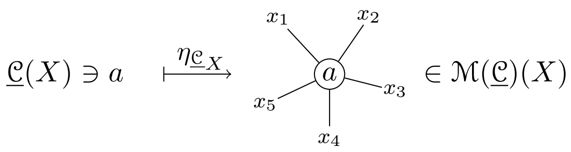
We now define α-*equivalence* on extended unrooted trees. Suppose first that
$$\mathcal{T} = \{a(x_1, \ldots, x_n), \ldots ; \sigma\}$$
is an ordinary unrooted tree, with $a \in \underline{\mathcal{C}}(X)$, $x_i \in FV(a)\backslash FV(\mathcal{T})$ and $\sigma(x_i) = y_j$. Let $\tau : X' \to X$ be a bijection that renames x_i to z, where z is fresh with respect to $V(\mathcal{T})\backslash\{x_i\}$. The α-equivalence (for ordinary unrooted trees) is the smallest equivalence relation generated by equalities
$$\{a(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n), \ldots ; \sigma\} =_\alpha \{a^\tau(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n), \ldots ; \sigma'\}, \tag{2.3.2}$$
where $\sigma' = \sigma$ on $V(\mathcal{T})\backslash\{x_i, y_j\}$ and $\sigma'(z) = y_j$. This definition generalises in a natural way to extended unrooted trees: to the set of generators given by (2.3.2), we add the clauses
$$\{(x, y), \ldots ; \sigma\} =_\alpha \{(x, z), \ldots ; \sigma'\},$$
where, for some variable $x_i \in V(\{(x, y), \ldots ; \sigma\})$, $\sigma(y) = x_i$ (i.e. y is not a free variable of $\{(x, y), \ldots ; \sigma\}$), z is fresh in the same sense as earlier, and σ' is the obvious modification of σ. In simple terms, we consider α-equivalent any two trees such that we can obtain one from another only by renaming variables which are not fixed points of the corresponding involutions.

We shall denote with $[\mathcal{T}]_\alpha$ the α-equivalence class determined by an (extended) unrooted tree $\mathcal{T}$. Finally, we shall denote with $\mathtt{T}_{\underline{\mathcal{C}}}(X)$ (resp. $\mathtt{eT}_{\underline{\mathcal{C}}}(X)$) the set of all α-equivalence classes of unrooted trees (resp. extended unrooted trees) whose parameters belong to $P_{\underline{\mathcal{C}}}$ and whose free variables are given by the set X. If X is a two-element set, this definition includes the possibility that an unrooted tree has 0 parameters, in which case the corresponding equivalence class is determined by the appropriate exceptional unrooted tree. We shall write $\mathtt{T}_{\underline{\mathcal{C}}}$ (resp. $\mathtt{eT}_{\underline{\mathcal{C}}}$) for the collection of all unrooted trees (resp. extended unrooted trees) generated by $P_{\underline{\mathcal{C}}}$.

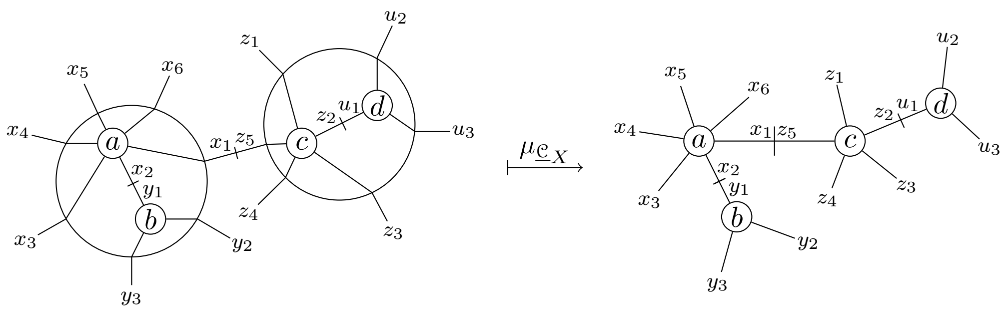
2.3.2 The monad of unrooted trees

The monad of unrooted trees is the monad $(\mathcal{M}, \mu, \eta)$ on the functor category $\mathbf{Set}^{\mathbf{Bij}^{op}}$, defined as follows. The endofunctor $\mathcal{M}$ is defined by
$$\mathcal{M}(\underline{\mathcal{C}})(X) = \mathtt{T}_{\underline{\mathcal{C}}}(X).$$

The component $\eta_{\underline{\mathcal{C}}_X} : \underline{\mathcal{C}}(X) \to \mathcal{M}(\underline{\mathcal{C}})(X)$ of the monad unit associates to $a \in \underline{\mathcal{C}}(X)$ the isomorphism class of the unrooted tree $\{a(x_1, \ldots, x_n), id_X\}$, where $X = \{x_1, \ldots, x_n\}$:



The action of the monad multiplication, typically (and incompletely) described as "flattening" in the literature, which could be imagined as illustrated in the picture below



deserves more attention if one wants to make a proper treatment of units of cyclic operads.

In order to obtain its complete description, we first build a rewriting system on $\mathtt{eT}_{\underline{\mathcal{C}}}$. The rewriting relation $\to$ on classes of $\mathtt{eT}_{\underline{\mathcal{C}}}$ is canonically induced by the reflexive and transitive closure of the union of the following reductions, defined on their representatives:

$$\{a(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n), (y, z), \ldots ; \sigma\} \to \{a^{\tau}(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n), \ldots ; \sigma'\}, \tag{2.3.3}$$

where $\sigma(x_i) = y$, τ renames x_i to z, and σ' is the obvious restriction of σ, and

$$\{(x, y), (u, v), \ldots ; \sigma\} \to \{(x, v), \ldots ; \sigma'\}, \tag{2.3.4}$$

where $\sigma(y) = u$, and σ' is again the obvious restriction of σ.

Lemma 2.13. *The rewriting system* $(\mathtt{eT}_{\underline{\mathcal{C}}}, \to)$ *is confluent and terminating.*

Proof. The termination of the system is obvious: in an arbitrary reduction sequence, each subsequent tree has one special corolla less, and the sequence finishes either when all of them are exhausted (in the case when the initial tree has at least one ordinary corolla), or when there is only one special corolla left (in the case when the initial tree consists only of special corollas). Due to the connectedness of unrooted trees, all special corollas (except one in the latter case) will indeed be exhausted. Clearly, the normal forms are precisely the unrooted trees of $\mathtt{T}_{\underline{\mathcal{C}}}$.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are reduced from $\mathcal{T}$ in one step, and if s_1 and s_2 are the special corollas involved in the respective reductions, the local confluence is proved by case analysis, with respect to whether s_1 and s_2 are equal or not. Let $c_1 \in Cor(\mathcal{T}_1)$ and $c_2 \in Cor(\mathcal{T}_2)$ be the corollas adjacent to s_1 and s_2, respectively, which were also affected by the reduction step, and let σ be the involution of $\mathcal{T}$.

- Suppose that $s_1 = s_2 = (x, y)$. Notice that, if $x, y \in FV(\mathcal{T})$, then (x, y) is the only corolla of $\mathcal{T}$, i.e. $\mathcal{T}$ is already a normal form. Also, if we have $\sigma(x) = x$, $\sigma(y) = x_i$ (or $\sigma(y) = y$, $\sigma(x) = x_i$), then $c_1 = c_2$, and $\mathcal{T}_1$ and $\mathcal{T}_2$ are trivially equal. Let us, therefore, assume that $x, y \notin FV(\mathcal{T})$. Let $\sigma(x) = x_i$ and $\sigma(y) = y_j$, where $x_i \in FV(c_1)$ and $y_j \in FV(c_2)$. Since extended unrooted trees contain no cycles, c_1 and c_2 must be different corollas. We proceed by analysing the shapes of c_1 and c_2.

– If $c_1 = a(x_1,\ldots,x_i,\ldots,x_n)$ and $c_2 = b(y_1,\ldots,y_j,\ldots,y_m)$, i.e. if

$$\mathcal{T} = \{a(x_1,\ldots,x_i,\ldots,x_n),(x,y),b(y_1,\ldots,y_j,\ldots,y_m),\ldots;\sigma\},$$

then both reductions arise by the reduction rule (2.3.3), leading to

$$\mathcal{T}_1 = \{a^{\tau_1}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n),b(y_1,\ldots,y_j,\ldots,y_m),\ldots;\sigma_1'\},$$

where $\sigma_1'(y)=y_j$, on one hand, and

$$\mathcal{T}_2 = (a(x_1,\ldots,x_i,\ldots,x_n),b^{\tau_2}(y_1,\ldots,y_{j-1},x,y_{j+1},y_m),\ldots;\sigma_2'\},$$

where $\sigma_2'(x)=x_i$, on the other hand. The trees $\mathcal{T}_1$ and $\mathcal{T}_2$ are clearly both α-equivalent with the tree

$$\mathcal{T}_3 = (a^{\tau_1}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n),b^{\tau_2}(y_1,\ldots,y_{j-1},x,y_{j+1},y_m),\ldots;\sigma_3'),$$

where $\sigma_3' = \sigma_1' = \sigma_2'$ on $V(\mathcal{T}_1)\backslash\{y,y_j\} = V(\mathcal{T}_2)\backslash\{x,x_i\}$ and $\sigma_3'(x) = y$. Therefore, $\mathcal{T}_1 =_\alpha \mathcal{T}_2$.

– Suppose now that $c_1 = a(x_1,\ldots,x_i,\ldots,x_n)$ and $c_2 = (y_j,z_j)$, i.e. that

$$\mathcal{T} = \{a(x_1,\ldots,x_i,\ldots,x_n),(x,y),(y_j,z_j),\ldots;\sigma\}.$$

In this case, by the reduction arising by (2.3.3), we get

$$\mathcal{T}_1 = \{a^{\tau}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n),(y_j,z_j),\ldots;\sigma_1'\},$$

where $\sigma_1'(y)=y_j$, and, by the reduction arising by (2.3.4), we get

$$\mathcal{T}_2 = \{a(x_1,\ldots,x_i,\ldots,x_n),(x,z_j),\ldots;\sigma_2'\},$$

where $\sigma_2'(x)=x_i$. The trees $\mathcal{T}_1$ and $\mathcal{T}_2$ can be reduced again, the respective reductions leading to

$$\mathcal{T}_1' = \{(a^{\tau})^{\tau_1}(x_1,\ldots,x_{i-1},z_j,x_{i+1},\ldots,x_n),\ldots;\sigma_1''\},$$

and

$$\mathcal{T}_2' = \{a^{\tau_2}(x_1,\ldots,x_{i-1},z_j,x_{i+1},\ldots,x_n),\ldots;\sigma_2''\}.$$

It is easy to verify that $\tau_1\circ\tau = \tau_2$ and $\sigma_1''=\sigma_2''$, from which we conclude that $\mathcal{T}_1'=\mathcal{T}_2'$, i.e., by the reflexivity of $=_\alpha$, that $\mathcal{T}_1' =_\alpha \mathcal{T}_2'$.

– If $c_1 = (w_i,x_i)$ and $c_2=(y_j,z_j)$, i.e. if

$$\mathcal{T} = \{(w_i,x_i),(x,y),(y_j,z_j),\ldots;\sigma\},$$

then

$$\mathcal{T}_1 = \{(w_i,y),(y_j,z_j),\ldots;\sigma_1'\},$$

with $\sigma_1(y)=y_j$, and

$$\mathcal{T}_2 = \{(w_i,x_i),(x,z_j),\ldots;\sigma_2'\},$$

with $\sigma_2'(x)=x_i$. The conclusion follows since, by the reduction rule (2.3.4), both $\mathcal{T}_1$ and $\mathcal{T}_2$ can now be reduced to the tree

$$\mathcal{T}_3 = \{(w_i,z_j),\ldots;\sigma_3'\}.$$

- If $s_1=(x,y)$ and $s_2=(u,v)$, we proceed by comparing c_1 and c_2, which now might be the same corollas.

– If $c_1 = c_2 = a(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$, and if $\sigma(x) = x_i$ and $\sigma(u) = x_j$, then the corresponding reductions of

$$\mathcal{T} = \{a(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n), (x, y), (u, v), \ldots ; \sigma\}$$

lead to

$$\mathcal{T}_1 = \{a^{\tau_1}(x_1, \ldots, y, \ldots, x_j, \ldots, x_n), (u, v), \ldots ; \sigma_1\}$$

and

$$\mathcal{T}_2 = (a^{\tau_2}(x_1, \ldots, x_i, \ldots, v, \ldots, x_n), (x, y), \ldots ; \sigma_2\},$$

where $\sigma_1(u) = x_j$ and $\sigma_2(x) = x_i$. This configuration of $\mathcal{T}_1$ and $\mathcal{T}_2$ is analogous to the one of the second item from the first case of the proof and the conclusion follows by the same argument.

– If $c_1 = c_2 = (z, w)$, $\sigma(x) = z$ and $\sigma(u) = w$, the reasoning is the same as in the third item from the first case of the proof.

– Finally, if $c_1 \neq c_2$, then $\mathcal{T}_1$ arises by the reduction involving c_1 and s_1, while c_2 and s_2 remain unchanged, and, symmetrically, $\mathcal{T}_2$ arises by the reduction involving c_2 and s_2, while c_1 and s_1 remain unchanged. By reducing $\mathcal{T}_1$ with respect to c_2 and s_2 and $\mathcal{T}_2$ with respect to c_2 and s_2, we clearly obtain the same tree.

■

By Lemma 2.13, an arbitrary normal form $nf(\mathcal{T})$ of an extended unrooted tree $\mathcal{T}$, with respect to the rewriting relation $\to$, determines a unique α-equivalence class $[nf(\mathcal{T})]_\alpha$ in $\mathtt{T}_{\underline{\mathcal{C}}}$. It is easily seen that, for every finite set X, this assignment gives rise to the function $nf_X : \mathtt{eT}_{\underline{\mathcal{C}}}(X) \to \mathtt{T}_{\underline{\mathcal{C}}}(X)$, determined by

$$nf_X : [\mathcal{T}]_\alpha \mapsto [nf(\mathcal{T})]_\alpha. \tag{2.3.5}$$

Next, we formally define the *flattening of two-level unrooted trees* (which is still not the monad multiplication), i.e. of the representatives of the isomorphism classes of

$$\mathcal{M}\mathcal{M}(\underline{\mathcal{C}})(X) = \mathcal{M}(\mathtt{T}_{\underline{\mathcal{C}}})(X) = \mathtt{T}_{\mathtt{T}_{\underline{\mathcal{C}}}}(X).$$

Observe that, syntactically, a two-level unrooted tree can be either

◇ an exceptional unrooted tree $\{(x, y); id_{\{x,y\}}\}$, or

◇ an ordinary unrooted tree

$$\{[\{a(x_1, x_2, \ldots), b(y_1, \ldots), \ldots ; \sigma_1\}]_\alpha(x_1, x_2, y_1, \ldots), \ldots, [\{(z_1, z_2); id_{\{z_1,z_2\}}\}]_\alpha(z_1, z_2), \ldots ; \sigma\},$$

whose parameters can be α-equivalence classes of both ordinary and exceptional unrooted trees of $\mathtt{T}_{\underline{\mathcal{C}}}$.

Remark 2.14. *Let $\mathcal{T}$ be a two-level unrooted tree. Suppose that, for $1 \leq i \leq n$, $[\mathcal{T}_i]_\alpha \in \mathtt{T}_{\underline{\mathcal{C}}}(Y_i)$ are the parameters of $\mathcal{T}$ and let C_i be their corresponding corollas. We then have $FV(C_i) = FV(\mathcal{T}_i) = Y_i$. The fact that the set of free variables of each corolla is recorded by the data of the corresponding parameter allows us to shorten the notation by writing $\mathcal{T}_i$ without listing explicitly the elements of $FV(\mathcal{T}_i)$. For example, for the tree from the latter case above, we shall write*

$$\{[\{a(x_1, x_2, \ldots), b(y_1, \ldots), \ldots ; \sigma_1\}]_\alpha, \ldots, [\{(z_1, z_2); id_{\{z_1,z_2\}}\}]_\alpha, \ldots ; \sigma\}.$$

We shall extend this abbreviation to trees of $\mathtt{eT}_{\mathtt{eT}_{\underline{\mathcal{C}}}}$, and when the form of the parameters of a two-level tree is irrelevant, we shall write $\{[\mathcal{T}_1]_\alpha, \ldots, [\mathcal{T}_n]_\alpha, s_1, \ldots, s_m; \sigma\}$, where s_i are special corollas.

The flattening of two-level unrooted trees is a familly of functions

$$flat_X : \mathtt{T}_{\mathtt{T}_{\underline{\mathcal{C}}}}(X) \to \mathtt{eT}_{\underline{\mathcal{C}}}(X),$$

indexed by finite sets, defined by the following two clauses:

- $\mathit{flat}_{\{x,y\}}([\{(x,y);id_{\{x,y\}}\}]_\alpha)=[\{(x,y);id_{\{x,y\}}\}]_\alpha$, and
- if $\mathcal{T}=\{[\{a(x_1,x_2,\dots),b(y_1,\dots),\dots;\sigma_1\}]_\alpha,\dots,[\{(z_1,z_2);id_{\{z_1,z_2\}}\}]_\alpha,\dots;\sigma\}$, then

$$\mathit{flat}_X([\mathcal{T}]_\alpha)=[\{a(x_1,x_2,\dots),b(y_1,\dots),\dots,(z_1,z_2),\dots;\underline{\sigma}\}]_\alpha,$$

  where, having denoted with $\mathcal{T}_i$, $1\leq i\leq n$, the corollas of $\mathcal{T}$, and with σ_i the corresponding involutions,

$$\underline{\sigma}(x)=\begin{cases}\sigma(x) & \text{if } x\in\bigcup_{i=1}^n FV(\mathcal{T}_i)\\ \sigma_i(x) & \text{if } x\in V(\mathcal{T}_i)\backslash FV(\mathcal{T}_i)\,.\end{cases}$$

Observe that α-equivalence of two-level unrooted trees also occurs on two levels. Therefore, a comment about the validness of the definition of flat_X is not superfluous. The proof of the following fact is nonetheless straightforward.

Lemma 2.15. *The function $\mathit{flat}_X:\mathtt{T}_{\mathtt{T}_{\underline{\mathcal{C}}}}(X)\rightarrow\mathtt{eT}_{\underline{\mathcal{C}}}(X)$ is well-defined.*

Observe that $\mathit{flat}_X([\mathcal{T}]_\alpha)$ is an α-equivalence class of an extended unrooted tree whenever $\mathcal{T}$ contains at least two corollas, one of which is special. These are the cases that make a gap between the flattening function and the action of the monad multiplication (which always results in an ordinary unrooted tree). In the same style as we presented the functions nf_X by (2.3.5), in what follows, we shall often denote the class $\mathit{flat}_X([\mathcal{T}]_\alpha)$ simply by $[\mathit{flat}(\mathcal{T})]_\alpha$.

The complete characterisation of the monad multiplication

$$\mu_{\underline{\mathcal{C}}}:\mathtt{T}_{\mathtt{T}_{\underline{\mathcal{C}}}}\rightarrow\mathtt{T}_{\underline{\mathcal{C}}}$$

is given by

$$\mu_{\underline{\mathcal{C}}X}=\mathit{nf}_X\circ\mathit{flat}_X.$$

Therefore, for $[\mathcal{T}]_\alpha\in\mathtt{T}_{\mathtt{T}_{\underline{\mathcal{C}}}}(X)$, we have

$$\mu_{\underline{\mathcal{C}}X}:[\mathcal{T}]_\alpha\mapsto[\mathit{nf}(\mathit{flat}(\mathcal{T}))]_\alpha.$$

Hence, in the presence of units, the action of the monad multiplication is indeed more than just "flattening", as commonly stated.

We now prepare the grounds for the proof that $(\mathcal{M},\mu,\eta)$ is indeed a monad.

We first extend naturally the domain of flattening to $\mathcal{M}'\mathcal{M}'(\underline{\mathcal{C}})$, where $\mathcal{M}'(\underline{\mathcal{C}})(X)=\mathtt{eT}_{\underline{\mathcal{C}}}(X)$. The clause that needs to be added to encompass the classes of $\mathtt{eT}_{\mathtt{eT}_{\underline{\mathcal{C}}}}(X)$ concerns two-level trees of the form

$$\{[\{a(x_1,x_2,\dots),b(y_1,\dots),(z_1,z_2)\dots;\sigma_1\}]_\alpha,\dots,[\{(u_1,u_2);id_{\{u_1,u_2\}}\}]_\alpha,\dots,(v_1,v_2),\dots;\sigma\},$$

i.e. extended unrooted trees whose set of corollas allows special corollas and the classes of extended unrooted trees. Let us denote with $\mathcal{T}$ the above tree, and let $Cor_s(\mathcal{T})$ be the set of its special corollas. The flattening of $[\mathcal{T}]_\alpha$ is defined simply as

$$\mathit{flat}_X([\mathcal{T}]_\alpha)=[\{a(x_1,x_2,\dots),b(y_1,\dots),(z_1,z_2),\dots,(u_1,u_2),\dots,(v_1,v_2),\dots;\underline{\sigma}\}]_\alpha,$$

with $\underline{\sigma}$ being defined exactly like before for the variables coming from $Cor(\mathcal{T})\backslash Cor_s(\mathcal{T})$, while we set $\underline{\sigma}(x)=\sigma(x)$ for all variables $x\in\bigcup_{s\in Cor_s(\mathcal{T})}FV(s)$.

For $[\mathcal{T}_1]_\alpha,[\mathcal{T}_2]_\alpha\in\mathtt{eT}_{\mathtt{eT}_{\underline{\mathcal{C}}}}(X)$, the following two lemmas give conditions which ensure that $\mathit{flat}_X([\mathcal{T}_1]_\alpha)\rightarrow\mathit{flat}_X([\mathcal{T}_2]_\alpha)$ in the rewriting system $(\mathtt{eT}_{\underline{\mathcal{C}}},\rightarrow)$.

Lemma 2.16. *For $[\mathcal{T}_1]_\alpha,[\mathcal{T}_2]_\alpha\in\mathtt{eT}_{\mathtt{eT}_{\underline{\mathcal{C}}}}(X)$, if $[\mathcal{T}_1]_\alpha\rightarrow[\mathcal{T}_2]_\alpha$ in $(\mathtt{eT}_{\mathtt{eT}_{\underline{\mathcal{C}}}},\rightarrow)$, then $\mathit{flat}_X([\mathcal{T}_1]_\alpha)\rightarrow\mathit{flat}_X([\mathcal{T}_2]_\alpha)$ in $(\mathtt{eT}_{\underline{\mathcal{C}}},\rightarrow)$.*

Proof. By case analysis relative to the shapes of corollas involved in the reduction $[\mathcal{T}_1]_\alpha \to [\mathcal{T}_2]_\alpha$. ∎

Lemma 2.17. *For* $[\{[\mathcal{T}_1]_\alpha, \dots, [\mathcal{T}_n]_\alpha, s_1, \dots, s_m; \sigma\}]_\alpha \in \mathtt{eT}_{\mathtt{eT}_{\underline{\mathcal{C}}}}(X)$ *and* $1 \le j \le n$, *if* $[\mathcal{T}_j]_\alpha \to [\mathcal{T}'_j]_\alpha$ *in* $(\mathtt{eT}_{\underline{\mathcal{C}}}, \to)$, *then*

$$\mathit{flat}_X([\{[\mathcal{T}_1]_\alpha, \dots, [\mathcal{T}_j]_\alpha, \dots, [\mathcal{T}_n]_\alpha, s_1, \dots, s_m; \sigma\}]_\alpha) \to \mathit{flat}_X([\{[\mathcal{T}_1]_\alpha, \dots, [\mathcal{T}'_j]_\alpha, \dots, [\mathcal{T}_n]_\alpha, s_1, \dots, s_m; \sigma\}]_\alpha)$$

in $(\mathtt{eT}_{\underline{\mathcal{C}}}, \to)$

Proof. By case analysis relative to the shapes of corollas involved in the reduction $[\mathcal{T}_j]_\alpha \to [\mathcal{T}'_j]_\alpha$. ∎

Relying on Lemma 2.16 and Lemma 2.17, we obtain the following two equivalent characterisations of the monad multiplication.

Lemma 2.18. *For* $[\mathcal{T}]_\alpha = [\{[\mathcal{T}_1]_\alpha, \dots, [\mathcal{T}_n]_\alpha, s_1, \dots, s_m; \sigma\}]_\alpha \in \mathtt{eT}_{\mathtt{eT}_{\underline{\mathcal{C}}}}(X)$ *the following claims hold:*

1. $\mathit{nf}(\mathit{flat}(\mathcal{T})) =_\alpha \mathit{nf}(\mathit{flat}(\mathit{nf}(\mathcal{T})))$,
2. $\mathit{nf}(\mathit{flat}(\mathcal{T})) =_\alpha \mathit{nf}(\mathit{flat}(\{[\mathit{nf}(\mathcal{T}_1)]_\alpha, \dots, [\mathit{nf}(\mathcal{T}_n)]_\alpha, s_1, \dots, s_m; \sigma\}))$.

Proof. By the termination of $(\mathtt{eT}_{\mathtt{eT}_{\underline{\mathcal{C}}}}, \to)$, we have that $\mathcal{T} \to \mathit{nf}(\mathcal{T})$, and then, by Lemma 2.16 and the termination of $(\mathtt{eT}_{\underline{\mathcal{C}}}, \to)$, we get the following sequence of reductions in $(\mathtt{eT}_{\mathcal{C}}, \to)$,

$$\mathit{flat}(\mathcal{T}) \to \mathit{flat}(\mathit{nf}(\mathcal{T})) \to \mathit{nf}(\mathit{flat}(\mathit{nf}(\mathcal{T}))).$$

On the other hand, by the termination of $(\mathtt{eT}_{\underline{\mathcal{C}}}, \to)$, we also have that $\mathit{flat}(\mathcal{T}) \to \mathit{nf}(\mathit{flat}(\mathcal{T}))$. Therefore, the first claim follows by the confluence of $(\mathtt{eT}_{\underline{\mathcal{C}}}, \to)$.

As for the second claim, by the termination of $(\mathtt{eT}_{\underline{\mathcal{C}}}, \to)$, we have $\mathcal{T}_i \to \mathit{nf}(\mathcal{T}_i)$, for all $i \in I$. Hence, by Lemma 2.17, and then again by the termination of $(\mathtt{eT}_{\underline{\mathcal{C}}}, \to)$, we get that

$$\begin{array}{rcl} \mathit{flat}(\mathcal{T}) & \to & \mathit{flat}(\{[\mathit{nf}(\mathcal{T}_1)]_\alpha, \dots, [\mathit{nf}(\mathcal{T}_n)]_\alpha, s_1, \dots, s_m; \sigma\}) \\ & \to & \mathit{nf}(\mathit{flat}(\{\mathit{nf}(\mathcal{T}_1), \dots, \mathit{nf}(\mathcal{T}_n), s_1, \dots, s_m; \sigma\})) \end{array}$$

is a reduction sequence of $(\mathtt{eT}_{\underline{\mathcal{C}}}, \to)$. The conclusion follows as in the previous claim. ∎

On the other hand, by the very definition of flattening on extended unrooted trees, we have the following property.

Lemma 2.19. *For* $\mathcal{T} = \{[\mathcal{T}_1]_\alpha, \dots, [\mathcal{T}_n]_\alpha; \sigma\} \in \mathtt{T}_{\mathtt{eT}_{\mathtt{eT}_{\underline{\mathcal{C}}}}}$, *the following equality holds:*

$$\mathit{flat}(\mathit{flat}(\mathcal{T})) = \mathit{flat}(\{[\mathit{flat}(\mathcal{T}_1)]_\alpha, \dots, [\mathit{flat}(\mathcal{T}_n)]_\alpha; \sigma\}).$$

We now finally verify the laws of the monad $(\mathcal{M}, \mu, \eta)$.

Lemma 2.20. *For natural transformations* $\mu : \mathcal{M}\mathcal{M} \to \mathcal{M}$ *and* $\eta : 1 \to \mathcal{M}$, *the following diagrams commute for every functor* $\underline{\mathcal{C}} : \mathbf{Bij}^{op} \to \mathbf{Set}$ *and finite set* X*:*

$$\begin{tikzcd} \mathcal{M}\mathcal{M}\mathcal{M}(\underline{\mathcal{C}})(X) \arrow[r, "\mathcal{M}\mu_{\underline{\mathcal{C}}_X}"] \arrow[d, "\mu\mathcal{M}_{\underline{\mathcal{C}}_X}"'] & \mathcal{M}\mathcal{M}(\underline{\mathcal{C}})(X) \arrow[d, "\mu_{\underline{\mathcal{C}}_X}"] \\ \mathcal{M}\mathcal{M}(\underline{\mathcal{C}})(X) \arrow[r, "\mu_{\underline{\mathcal{C}}_X}"] & \mathcal{M}(\underline{\mathcal{C}})(X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}(\underline{\mathcal{C}})(X) \arrow[r, "\mathcal{M}\eta_{\underline{\mathcal{C}}_X}"] \arrow[dr, "id_{\underline{\mathcal{C}}_X}"'] & \mathcal{M}\mathcal{M}(\underline{\mathcal{C}})(X) \arrow[d, "\mu_{\underline{\mathcal{C}}_X}"] \\ & \mathcal{M}(\underline{\mathcal{C}})(X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}(\underline{\mathcal{C}})(X) \arrow[r, "\eta\mathcal{M}_{\underline{\mathcal{C}}_X}"] \arrow[dr, "id_{\underline{\mathcal{C}}_X}"'] & \mathcal{M}\mathcal{M}(\underline{\mathcal{C}})(X) \arrow[d, "\mu_{\underline{\mathcal{C}}_X}"] \\ & \mathcal{M}(\underline{\mathcal{C}})(X) \end{tikzcd}$$

Proof. We begin with the left diagram. Chasing the associativity of multiplication includes treating several cases, relative to the shape of the unrooted tree of

$$\mathcal{MMM}(\underline{\mathcal{C}})(X) = \mathtt{T}_{\mathtt{T}_{\mathtt{T}_{\underline{\mathcal{C}}}}}(X)$$

that we start from. The most interesting is the one starting from (a class determined by) an ordinary unrooted tree with corollas given by ordinary unrooted trees built over $\mathtt{T}_{\underline{\mathcal{C}}}$ and we prove the associativity only for this case. Let, therefore, $\mathcal{T} = \{[\mathcal{T}_1]_\alpha, \ldots, [\mathcal{T}_n]_\alpha; \sigma\}$.

By chasing the diagram to the right-down, the action of $\mathcal{M}\mu_{\underline{\mathcal{C}}_X}$ corresponds to corolla-per-corolla flattening of $\mathcal{T}$, followed by taking the respective normal forms. Then μ flattens additionally the resulting tree and reduces it to a normal form. These actions make the following sequence of steps:

$$\begin{array}{rcl}\mathcal{T} & \mapsto & \{[\mathit{flat}(\mathcal{T}_1)]_\alpha, \ldots, [\mathit{flat}(\mathcal{T}_n)]_\alpha; \sigma\} \\ & \mapsto & \{[\mathit{nf}(\mathit{flat}(\mathcal{T}_1))]_\alpha, \ldots, [\mathit{nf}(\mathit{flat}(\mathcal{T}_n))]_\alpha; \sigma\} \\ & \mapsto & \mathit{flat}(\{[\mathit{nf}(\mathit{flat}(\mathcal{T}_1))]_\alpha, \ldots, [\mathit{nf}(\mathit{flat}(\mathcal{T}_n))]_\alpha; \sigma\}) \\ & \mapsto & \mathit{nf}(\mathit{flat}(\{[\mathit{nf}(\mathit{flat}(\mathcal{T}_1))]_\alpha, \ldots, [\mathit{nf}(\mathit{flat}(\mathcal{T}_n))]_\alpha; \sigma\})) = R.\end{array}$$

The action $\mu\mathcal{M}_{\underline{\mathcal{C}}_X}$ on the left-down side of the diagram corresponds to the action of μ on the tree $\mathcal{T}$ in whole, which flattens it and reduces it to a normal form. Followed by μ again, this gives us the following sequence:

$$\begin{array}{rcl}\mathcal{T} & \mapsto & \mathit{flat}(\mathcal{T}) \\ & \mapsto & \mathit{nf}(\mathit{flat}(\mathcal{T})) \\ & \mapsto & \mathit{flat}(\mathit{nf}(\mathit{flat}(\mathcal{T}))) \\ & \mapsto & \mathit{nf}(\mathit{flat}(\mathit{nf}(\mathit{flat}(\mathcal{T})))) = L.\end{array}$$

Let $R' = \mathit{nf}(\mathit{flat}(\{[\mathit{flat}(\mathcal{T}_1)]_\alpha, \ldots, [\mathit{flat}(\mathcal{T}_n)]_\alpha; \sigma\}))$ and $L' = \mathit{nf}(\mathit{flat}(\mathit{flat}(\mathcal{T})))$. By Lemma 2.18, we have that $R = R'$ and $L = L'$, and, by Lemma 2.19, we have that $R' = L'$.

We now verify the unit laws for the case when $[\mathcal{T}]_\alpha \in \mathcal{M}(\underline{\mathcal{C}})(X)$ is determined by an ordinary unrooted tree. Let, therefore, $\mathcal{T} = \{a_1(x_1, \ldots, x_k), \ldots, a_n(y_1, \ldots, y_r); \sigma\}$.

By going to the right-down in the first unit diagram (i.e. the diagram in the middle), the action of $\mathcal{M}_{\eta_{\underline{\mathcal{C}}_X}}$ turns each corolla a_i into a single-corolla unrooted tree $\mathcal{T}_i$, leading to a two-level unrooted tree, which is then flattened and reduced to a normal form by μ. Therefore, the right-down side sequence is as follows:

$$\begin{array}{rcl}\mathcal{T} & \mapsto & \{[\{a_1(x_1, \ldots, x_k), id\}]_\alpha, \ldots, [\{a_n(y_1, \ldots, y_r); id\}]_\alpha; \sigma\} \\ & \mapsto & \{a_1(x_1, \ldots, x_k), \ldots, a_n(y_1, \ldots, y_r); \underline{\sigma}\} \\ & \mapsto & \{a_1(x_1, \ldots, x_k), \ldots, a_n(y_1, \ldots, y_r); \underline{\sigma}'\}\end{array}$$

the resulting tree being exactly $\mathcal{T}$, since

$$\underline{\sigma}'(x) = \underline{\sigma}(x) = \begin{cases} \sigma(x) & \text{if } x \in \bigcup_{i=1}^n FV(\mathcal{T}_i) \\ x & \text{if } x \in V(\mathcal{T}_i)\backslash FV(\mathcal{T}_i) \end{cases} = \begin{cases} \sigma(x) & \text{if } x \in V(\mathcal{T}) \\ x & \text{if } x \in V(\mathcal{T}_i)\backslash FV(\mathcal{T}_i) \end{cases} = \sigma(x),$$

wherein the last equality holds since $V(\mathcal{T}_i)\backslash FV(\mathcal{T}_i) = \emptyset$, for all $1 \le i \le n$.

By chasing the second unit diagram to the right, $\mathcal{T}$ will first be turned, by the action of $\eta\mathcal{M}_{\underline{\mathcal{C}}_X}$, into a single-corolla two-level tree, which will then be flattened and reduced to a normal form by the action of μ. Therefore, we have the sequence

$$\begin{array}{rcl}\mathcal{T} & \mapsto & \{[\{a_1(x_1, \ldots, x_k), \ldots, a_n(y_1, \ldots, y_r); \sigma\}]_\alpha, id_X\} \\ & \mapsto & \{a_1(x_1, \ldots, x_k), \ldots, a_n(y_1, \ldots, y_r); \underline{id_X}\} \\ & \mapsto & \{a_1(x_1, \ldots, x_k), \ldots, a_n(y_1, \ldots, y_r); \underline{id_X}'\}\end{array}$$

For the resulting involution $\underline{id_X}'$ we have

$$\underline{id_X}'(x) = \underline{id_X}(x) = \begin{cases} x & \text{if } x \in FV(\mathcal{T}) \\ \sigma(x) & \text{if } x \in V(\mathcal{T})\backslash FV(\mathcal{T}) \end{cases} = \sigma(x).$$

Therefore, the resulting tree is exactly $\mathcal{T}$. ■

Finally, here is the original definition [GK95, Definition 2.1] of a cyclic operad, recasted in the new syntactic framework.

Definition 2.21. A cyclic operad is an algebra over the monad $(\mathcal{M}, \mu, \eta)$.

And, under these syntactic glasses, the well-known result about the equivalence of the biased and unbiased definitions tells that *a functor* $\underline{\mathcal{C}} : \mathbf{Bij}^{op} \to \mathbf{Set}$ *carries a cyclic operad structure (as described by Definition 1.4) if and only if it is endowed with a structure morphism of an* $\mathcal{M}$*-algebra structure on* $\underline{\mathcal{C}}$. Formally, denoting with $\mathbf{Alg}_{\mathcal{M}}(\mathbf{Set}^{\mathbf{Bij}^{op}})$ the category of $\mathcal{M}$-algebras in $\mathbf{Set}^{\mathbf{Bij}^{op}}$, the following theorem holds.

Theorem 2.22. *The categories* $\mathbf{CO_{en}}$ *and* $\mathbf{Alg}_{\mathcal{M}}(\mathbf{Set}^{\mathbf{Bij}^{op}})$ *are isomorphic.*

Before we introduce the μ-syntax in the following section, and ultimatelly prove Theorem 2.22, we indicate the biased cyclic operad structure "hiding" in the monad approach we just formalised. As we shall see, the exceptional unrooted trees will be used as pasting schemes of identities of cyclic operads. The reason to go for exceptional unrooted trees (instead of taking explicit corollas, i.e. parameters, for identities), is that, otherwise, in the definition of the free cyclic operad over $P_{\underline{\mathcal{C}}}$ that follows, $\mathtt{eT}_{\underline{\mathcal{C}}}(X)$ would have to be quotiented by more than the corolla-preserving isomorphisms, as the validation of the unit laws would involve the identification of trees which are not α-equivalent.

2.3.3 The free cyclic operad structure implicit in $(\mathcal{M}, \mu, \eta)$

In the unbiased approach of §2.3.2, the monad $\mathcal{M}$ actually arises from the adjunction $F \vdash U$, where $F : \mathbf{Set}^{\mathbf{Bij}^{op}} \to \mathbf{CO_{en}}$ is the free cyclic operad functor and $U : \mathbf{CO_{en}} \to \mathbf{Set}^{\mathbf{Bij}^{op}}$ is the obvious forgetful functor. For a functor $\underline{\mathcal{C}}$, we now reveal the definition of the free cyclic operad over $\underline{\mathcal{C}}$, which was implicit in §2.3.2.

The functor $F(\underline{\mathcal{C}}) : \mathbf{Bij}^{op} \to \mathbf{Set}$, underlying the free cyclic operad, is defined by $F(\underline{\mathcal{C}})(X) = \mathtt{T}_{\underline{\mathcal{C}}}(X)$. Before we spell out the biased free cyclic operad structure in the style of Definition 1.4, we fix some notation.

Notation 2.23. *For an unrooted tree* $\mathcal{T}$*, a finite set* V *and a bijection* $\vartheta : V \to V(\mathcal{T})$*, we shall denote with* $\mathcal{T}^{\vartheta}$ *the unrooted tree obtained from* $\mathcal{T}$ *by renaming its variables in a way dictated by* ϑ *and adapting its corollas accordingly. More precisely, if* $a \in Cor(\mathcal{T})$ *is an ordinary corolla,* $\mathcal{T}^{\vartheta}$ *will, instead of* a*, contain the corolla* $a^{\vartheta|^{FV(a)}}$*, and, if* $(x, y) \in Cor(\mathcal{T})$ *is a special corolla,* $\mathcal{T}^{\vartheta}$ *will, instead of* (x, y)*, contain the corolla* $(\vartheta^{-1}(x), \vartheta^{-1}(y))$*. The involution* σ^{ϑ} *of* $\mathcal{T}^{\vartheta}$ *is defined as* $\sigma^{\vartheta}(v) = \vartheta^{-1}(\sigma(\vartheta(v)))$*, for* $v \in V$*.*

For a bijection $\kappa : X' \to X$, the image $[\mathcal{T}]^{\kappa}_{\alpha}$ of $[\mathcal{T}]_{\alpha} \in \mathtt{T}_{\underline{\mathcal{C}}}(X)$ under $\mathtt{T}_{\underline{\mathcal{C}}}(\kappa) : \mathtt{T}_{\underline{\mathcal{C}}}(X) \to \mathtt{T}_{\underline{\mathcal{C}}}(X')$ is the equivalence class $[\mathcal{T}^{\kappa \cup \varepsilon}]_{\alpha}$, where $\varepsilon : V \to V(\mathcal{T})\backslash X$ is an arbitrary bijection, such that $X' \cap V = \emptyset$.

Let X and Y be non-empty finite sets such that for some $x \in X$ and $y \in Y$ we have $X\backslash\{x\} \cap Y\backslash\{y\} = \emptyset$, and let $[\mathcal{T}_1]_{\alpha} \in \mathtt{T}_{\underline{\mathcal{C}}}(X)$, $[\mathcal{T}_2]_{\alpha} \in \mathtt{T}_{\underline{\mathcal{C}}}(Y)$. The partial composition operation

$${}_{x}\bullet_{y} \,:\, \mathtt{T}_{\underline{\mathcal{C}}}(X) \times \mathtt{T}_{\underline{\mathcal{C}}}(Y) \to \mathtt{T}_{\underline{\mathcal{C}}}(X\backslash\{x\} \cup Y\backslash\{y\})$$

is given as

$$[\mathcal{T}_1]_{\alpha} \, {}_{x}\bullet_{y} \, [\mathcal{T}_2]_{\alpha} = [nf(\mathcal{T})]_{\alpha},$$

where $Cor(\mathcal{T})$ is obtained by taking the union of the sets of corollas of $\mathcal{T}_1$ and $\mathcal{T}_2$, after having previously adapted them in a way that makes this union disjoint with respect to the variables occuring in it. More precisely, if $\vartheta_1 : V_1 \to (V(\mathcal{T}_1)\backslash X) \cup \{x\}$ and $\vartheta_2 : V_2 \to (V(\mathcal{T}_2)\backslash Y) \cup \{y\}$ are bijections such that $V_1 \cap V_2 = \emptyset$, then

$$Cor(\mathcal{T}) = \{C^{(\vartheta_1 \cup id_{X\backslash\{x\}})|^{FV(C)}} \mid C \in Cor(\mathcal{T}_1)\} \cup \{D^{(\vartheta_2 \cup id_{Y\backslash\{y\}})|^{FV(D)}} \mid D \in Cor(\mathcal{T}_2)\}.$$

If σ_i is the involution of $\mathcal{T}_i$, $i = 1, 2$, the involution σ of $\mathcal{T}$ is defined as follows:

$$\sigma(v) = \begin{cases} \vartheta_1^{-1}(\sigma_1(\vartheta_1(v))) & \text{if } v \in V_1\backslash\vartheta_1^{-1}(x) \\ \vartheta_2^{-1}(y) & \text{if } v = \vartheta_1^{-1}(x) \\ \vartheta_2^{-1}(\sigma_2(\vartheta_2(v))) & \text{if } v \in \vartheta_2^{-1}(y) \\ \vartheta_1^{-1}(x) & \text{if } v = \vartheta_2^{-1}(y) \\ v & \text{if } v \in X\backslash\{x\} \cup Y\backslash\{y\}\,. \end{cases}$$

For an arbitrary two-element set $\{y, z\}$, we set $id_{y,z} = [\{(y, z); id_{\{y,z\}}\}]_\alpha$.

Remark 2.24. *Observe that the partial composition structure on classes of unrooted trees does not violate the constant-freeness requirement. The condition $\underline{\mathcal{C}}(\{x\}) = \emptyset$ plays an indispensable role here: requiring that $\underline{\mathcal{C}}(\emptyset) = \emptyset$, but allowing the possibility that $\underline{\mathcal{C}}(\{x\}) \neq \emptyset$, is inconsistent. Consider, for example, $a \in \underline{\mathcal{C}}(\{x\})$, $b \in \underline{\mathcal{C}}(\{y\})$ and their composition $[a(x)]_\alpha\, {}_x\!\bullet_y\, [b(y)]_\alpha$.*

2.4 μ-syntax

Backed up with the graphical intuition of the free cyclic operad structure on classes of unrooted trees described in §2.3.3, in this section we introduce the μ-syntax.

2.4.1 The language and the equations

For a functor $\underline{\mathcal{C}} : \mathbf{Bij}^{op} \to \mathbf{Set}$, such that $\underline{\mathcal{C}}(\emptyset) = \underline{\mathcal{C}}(\{x\}) = \emptyset$, the language of the μ-syntax is built over the collection of parameters $P_{\underline{\mathcal{C}}}$ (see (2.2.2)) and the set of variables $\mathtt{V}$. Unlike the combinator syntax $\mathtt{cTerm}_{\underline{\mathcal{C}}}$ specified in §2.2, which has only one kind of expressions, the μ-syntax features two different kinds of typed expressions, as shown in Figure 2.2, where $a \in P_{\underline{\mathcal{C}}}$ and $x \in \mathtt{V}$.

COMMANDS	TERMS
$c ::= \langle s \mid t \rangle \;\mid\; \underline{a}\{t_1, \ldots, t_n\}$	$s, t ::= x \;\mid\; \mu x.c$

FIGURE 2.2: Commands and terms of the μ-syntax

We denote the typing judgments for commands and terms with $c : X$ and $X \mid s$, respectively, where X ranges over finite sets. In expressions $c : X$ and $X \mid s$, the set X is the type of the command c and of the term s, respectively, and the backward typing judgment $X \mid s$ is used merely to further distinguish the representation of terms and commands. We say that $\underline{a}$ is the head symbol of a command $\underline{a}\{t_1, \ldots, t_n\}$.

The assignment of types to commands and terms is done by the rules listed in Figure 2.3.

$$\frac{}{\{x\} \mid x} \qquad \frac{a \in \underline{\mathcal{C}}(\{x_1, \ldots, x_n\}) \quad Y_i \text{ pairwise disjoint for all } i \in \{1, \ldots, n\} \quad Y_i \mid t_i \text{ for all } i \in \{1, \ldots, n\}}{\underline{a}\{t_1, \ldots, t_n\} : \bigcup_{i=1}^n Y_i} \qquad \frac{X \cap Y = \emptyset \quad X \mid s \quad Y \mid t}{\langle s \mid t \rangle : X \cup Y} \qquad \frac{c : X \quad x \in X}{X\backslash\{x\} \mid \mu x.c}$$

FIGURE 2.3: Typing rules of the μ-syntax

Remark 2.25. *Observe that, thanks to the disjointness assumptions in the two rules for typing commands, for each term $\mu x.c$, where $c : X$, the variable x bound by μ has a unique occurence among the variables of X.*

Intuitively, commands mimick operations of the free cyclic operad over the functor $\underline{\mathcal{C}}$, and, thereby, a judgement $c : X$ should be thought of as an unrooted tree whose free variables are precisely the elements of X. On the other hand, terms represent operations with one selected entry and the role of the set X in a judgement $X \,|\, s$ is to label all entries *except* the selected one. From the tree-wise perspective, this is represented by an unrooted tree whose set of free variables is $X \cup \{x\}$, where x is precisely the variable bound by μ (i.e. the variable placed immediatelly on the right of the symbol μ).

Remark 2.26. *It is easily seen that, for any command $c : X$, the set X contains at least two elements. Related to this is the fact that an expression of the form $\mu y.(\mu x.c)$ is not allowed by the typing rules. This reflects the constant-freeness requirement we imposed for cyclic operads.*

Notation 2.27. *We shall sometimes denote the commands introduced by the second typing rule as $\underline{a}\{t_x \,|\, x \in X\}$ (for $a \in \underline{\mathcal{C}}(X)$), or as $\underline{a}\{\sigma\}$, where σ assigns to every $x \in X$ a term t_x. The order of appearance of the t_x's in $\underline{a}\{t_x \,|\, x \in X\}$ is irrelevant. Whenever we use the notation, say $\underline{a}\{t, s\}$, for $a \in \underline{\mathcal{C}}(\{x, y\})$, it will be clear from the context whether we mean $\underline{a}\{t, s\} = \underline{a}\{\sigma\}$, with $\sigma(x) = t$ and $\sigma(y) = s$, or with σ defined in the other way around.*

The way commands are constructed is motivated by the action of the simultaneous and partial grafting of unrooted trees, formally defined through the composition operation ${}_x\bullet_y$ from §2.3.3. The command $\underline{a}\{t_x \,|\, x \in X\}$, introduced by the second rule, should be imagined as the simultaneous grafting of the corolla a and the "surrounding" trees t_x, one for each free variable x of a, along the variables bound by μ in each t_x. In the special case when, for some $x \in X$, the corresponding term t_x is a variable, say u, this process of grafting reduces to the renaming of the variable x of the corolla a to u. Therefore, if all the terms corresponding, by the second typing rule in Figure 2.3, to the elements of X are variables from the set, say, $V = \{u, v, w, ...\}$, then the appropriate command is $\underline{a}\{\sigma\}$, where $\sigma : V \to X$ is the bijection determined by that correspondence, and it describes the unrooted tree $\{a^\sigma(u, v, w, \dots); id_V\}$. The command $\langle s \,|\, t\rangle$ describes the grafting of unrooted trees represented by the terms s and t along their variables bound by μ. Therefore, the pattern $\langle \mu x.\_ \,|\mu y.\_\,\rangle$ corresponds to the composition $(-)_x\bullet_y(-)$ on classes of unrooted trees.

The equations of the μ-syntax are given in Figure 2.4.

(MU1)	$\langle s \,|\, t\rangle = \langle t \,|\, s\rangle$	(MU3)	$\mu x.c = \mu y.c[y/x]$
(MU2)	$\langle \mu x.c \,|\, s\rangle = c[s/x]$	(MU4)	$\underline{a}\{t_x \,|\, x \in X\} = \underline{a^\sigma}\{t_{\sigma(y)} \,|\, y \in Y\}$

FIGURE 2.4: The equations of the μ-syntax

In Figure 2.4, in (MU2), $c[s/x]$ denotes the command c in which the unique occurrence of the variable x in X (see Remark 2.25) has been replaced by the term s, in (MU3), y is fresh with respect to all variables of c except x, and, in (MU4), $\sigma : Y \to X$ is an arbitrary bijection.

The equation (MU1) stipulates the symmetry of grafting of unrooted trees, i.e. the commutativity of composition operations ${}_x\bullet_y$.

The equations (MU3) and (MU4) are α-conversions, capturing the same intuition as does α-conversion in λ-calculus. Intuitively, α-conversion tells that the name of the entry selected for the composition does not matter, which reflects the equivariance of composition operations ${}_x\bullet_y$. Put in an very simple context, α-conversion tells that the function $f(x)$ is the same as the function $f(y)$.

The substitution $c[s/x]$, figuring in the equation (MU2) (as well as the substitution $c[y/x]$ from (MU3)), must be performed in the *capture-avoiding* manner. This means that the variables which were originally "free" (i.e. *not bound* by μ) in c cannot become "captured" (i.e. *bound* by μ) after the substitution is made. This is achieved by renaming, prior to the substitution, all the bound variables in c and s, so that they are all turned mutually distinct, and then performing the appropriate substitution. For example,

$$\mu x.\underline{a}\{x,y\}[x/y] \neq \mu x.\underline{a}\{x,x\}, \quad \text{but} \quad \mu x.\underline{a}\{x,y\}[x/y] = \mu z.\underline{a^\sigma}\{z,y\}[x/y] = \mu z.\underline{a^\sigma}\{z,x\},$$

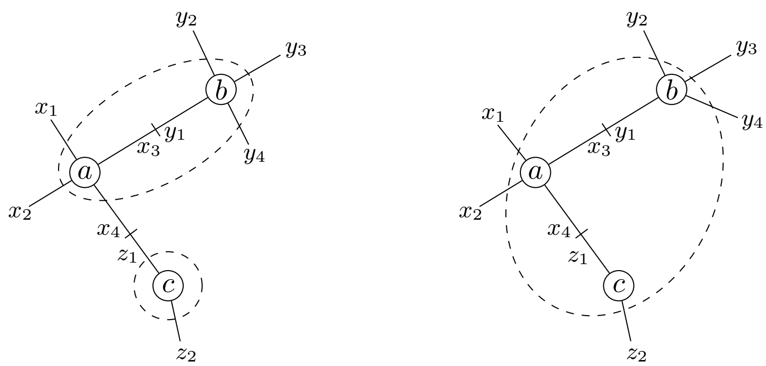
where σ renames x to z.

The equation (MU2) is quite evidently reminescent of the β-reduction of λ-calculus, when considered as a rewriting rule $\langle \mu x.c \,|\, s\rangle \to c[s/x]$, and it essentially captures the same idea of function application as λ-calculus. The intuition becomes more tangible from the point of view of trees: the commands $\langle \mu x.c \,|\, s\rangle$ and $c[s/x]$, equated with (MU2), describe two ways to build (by means of grafting) the same unrooted tree. Here is an example.

Example 2.28. Consider the unrooted tree

$$\mathcal{T} = \{a(x_1,x_2,x_3,x_4), b(y_1,y_2,y_3,y_4), c(z_1,z_2); \sigma\},$$

where $\sigma = (x_3\ \ y_1)(x_4\ \ z_1)$. One way to build $\mathcal{T}$ is to graft along x_4 and z_1 unrooted trees $\mathcal{T}_1 = \{a(x_1,x_2,x_3,x_4), b(y_1,y_2,y_3,y_4); \sigma_1\}$, where $\sigma_1 = (x_3\ \ y_1)$, and $\mathcal{T}_2 = \{c(z_1,z_2); id_{\{z_1,z_2\}}\}$, singled out with dashed lines in the left picture below:



The unrooted tree $\mathcal{T}_1$ (in the upper part of the left picture) can itself be seen as a grafting, namely the simultaneous grafting of the corolla a and its surrounding trees: in this case this involves explicit grafting only with the corolla b (along the free variables x_3 and y_1). This way of constructing $\mathcal{T}$ is described by the command

$$\langle \mu x_4.\underline{a}\{x_1,x_2,\mu y_1.\underline{b}\{y_1,y_2,y_3,y_4\},x_4\} \,|\, \mu z_1.\underline{c}\{z_1,z_2\}\rangle \tag{*}$$

that witnesses the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are connected along their selected free variables x_4 and z_1, respectively: x_4 and z_1 are bound with μ in the terms corresponding to these two trees. The subterm $\underline{a}\{x_1,x_2,\mu y_1.\underline{b}\{y_1,y_2,y_3,y_4\},x_4\}$ on the left-hand side is the command that accounts for the simultaneous grafting of the corolla a and its surrounding trees, while $\underline{c}\{z_1,z_2\}$ on the right-hand side stands for the corolla c. On the other hand, we could have chosen to build the tree $\mathcal{T}$ simply by making the simultaneous grafting of the corolla a and its surrounding trees, as indicated on the picture on the right. This way of building $\mathcal{T}$ is described with the command $\underline{a}\{x_1,x_2,\mu y_1.\underline{b}\{y_1,y_2,y_3,y_4\},\mu z_1.\underline{c}\{z_1,z_2\}\}$, which is, up to substitution, exactly the command

$$\underline{a}\{x_1,x_2,\mu y_1.\underline{b}\{y_1,y_2,y_3,y_4\},x_4\}[\mu z_1.\underline{c}\{z_1,z_2\}/x_4]$$

to which (*) reduces by applying the rewriting rule $\langle \mu x.c \,|\, s\rangle \to c[s/x]$. □

We shall denote with $\mu\mathtt{Exp}_{\underline{\mathcal{C}}}$ the set of all expressions of the μ-syntax induced by $P_{\underline{\mathcal{C}}}$, and we shall use $\mu\mathtt{Term}_{\underline{\mathcal{C}}}$ and $\mu\mathtt{Comm}_{\underline{\mathcal{C}}}$ to denote the subsets of terms and commands of $\mu\mathtt{Exp}_{\underline{\mathcal{C}}}$, respectively. As in the case of the combinator syntax, the set of expressions (resp. terms and commands) of type X will be denoted by $\mu\mathtt{Exp}_{\underline{\mathcal{C}}}(X)$ (resp. $\mu\mathtt{Term}_{\underline{\mathcal{C}}}(X)$ and $\mu\mathtt{Comm}_{\underline{\mathcal{C}}}(X)$).

2.4.2 μ-syntax as a rewriting system

Let $\rightsquigarrow$ be the rewriting relation defined on $\mu\mathtt{Exp}_{\underline{\mathcal{C}}}$ as the reflexive and transitive closure of the union of rewriting rules

$$\langle s \,|\, t\rangle \rightsquigarrow \langle t \,|\, s\rangle \quad \text{and} \quad \langle \mu x.c \,|\, s\rangle \rightsquigarrow c[s/x]$$

obtained by orienting from left to right the equations (MU1) and (MU2), respectively, which is, moreover, congruent with respect to (MU3), (MU4) and substitution[1].

The non-confluence of the rewriting system $(\mu\mathtt{Exp}_{\underline{\mathcal{C}}}, \rightsquigarrow)$ shows up immediately: for reductions

$$c_2[\mu x.c_1/y] \leftsquigarrow \langle \mu x.c_1 \,|\, \mu y.c_2\rangle \rightsquigarrow c_1[\mu y.c_2/x]$$

arising due to (MU1) (which makes the whole reduction system symmetric), we do not have a way to exhibit a command c, such that $c_2[\mu x.c_1/y] \rightsquigarrow c$ and $c_1[\mu y.c_2/x] \rightsquigarrow c$. Nevertheless, all three commands above describe the same unrooted tree.

However, modulo the trivial commuting conversion, this rewriting system is terminating: as a consequence of the fact that an element $x \in X$ appears only once in a command $c : X$, the number of μ-binders in an expression is strictly decreasing at each reduction step of the form $\langle \mu x.c \,|\, s\rangle \rightsquigarrow c[s/x]$. It is straightforward to prove that the set $\mu\mathtt{Exp}^{nf}_{\underline{\mathcal{C}}} = \mu\mathtt{Comm}^{nf}_{\underline{\mathcal{C}}} \cup \mu\mathtt{Term}^{nf}_{\underline{\mathcal{C}}}$ of normal forms is generated by the following rules:

$$\frac{}{x \in \mu\mathtt{Term}^{nf}_{\underline{\mathcal{C}}}} \qquad \frac{a \in \underline{\mathcal{C}}(X) \qquad t_x \in \mu\mathtt{Term}^{nf}_{\underline{\mathcal{C}}} \text{ for all } x \in X}{\underline{a}\{t_x \,|\, x \in X\} \in \mu\mathtt{Comm}^{nf}_{\underline{\mathcal{C}}}} \qquad \frac{c \in \mu\mathtt{Comm}^{nf}_{\underline{\mathcal{C}}}}{\mu x.c \in \mu\mathtt{Term}^{nf}_{\underline{\mathcal{C}}}}$$

In the next example, we examine the shape of normal forms in relation with unrooted trees.

Example 2.29. Let $\mathcal{T}$ be the unrooted tree from Example 2.28. Here is the list of commands in normal form that describe $\mathcal{T}$:

$$\underline{a}\{x_1, x_2, \mu y_1.\underline{b}\{y_1, y_2, y_3, y_4\}, \mu z_1.\underline{c}\{z_1, z_2\}\},$$

$$\underline{b}\{\mu x_3.\underline{a}\{x_1, x_2, x_3, \mu z_1.\underline{c}\{z_1, z_2\}\}, y_2, y_3, y_4\}, \quad \underline{b}\{\mu x_3.\underline{c}\{\mu x_4.\underline{a}\{x_1, x_2, x_3, x_4\}, z_2\}, y_2, y_3, y_4\},$$

$$\underline{c}\{\mu x_4.\underline{a}\{\mu y_1.\underline{b}\{y_1, y_2, y_3, y_4\}, x_2, x_3, x_4\}, z_2\}, \quad \underline{c}\{\mu x_4.\underline{b}\{\mu x_3.\underline{a}\{x_1, x_2, x_3, x_4\}, y_2, y_3, y_4\}, z_2\}.$$

Each of the commands records the free variables and corollas of $\mathcal{T}$: free variables are the variables not bound with μ (x_1, x_2, y_2, y_3, y_4 and z_2), and the corollas correspond to the underlined parameters (a, b and c). The variables involved in edges of $\mathcal{T}$ (x_3, y_1, x_4 and z_1) can also be recovered from the list, as the variables bound with μ. For example, in the first command we see that y_1 and z_1 are explicitly bound by μ, while for x_3 and x_4 we could say that they are implicitly bound, given that they are replaced with a non-variable term.

In general, the set $\mu\mathtt{Comm}^{nf}_{\underline{\mathcal{C}}}$ describes decompositions of unrooted trees of the following kind: pick a corolla a of a tree, and then proceed recursively so in all the connected components of the graph resulting from the removal of a. (We provide in §2.4.4 an algorithmic computation of

[1] Since the precautionary renaming which ensures that the substitution $c[s/x]$ is done in the capture-free manner is *non-deterministinc*, the rewriting relation $\rightsquigarrow$ is formally defined on the equivalence classes of expressions of the μ-syntax with respect to (MU3) and (MU4), just as the usual rewriting systems in λ-calculus are actually defined on α-conversion classes.

these connected components).

Amusingly, one can show that, if (MU1) gets oriented in the other way around, the normal forms of the resulting rewriting system will be in one-to-one correspondence with the combinators of Section 2.2, and thus describe decompositions of unrooted trees of the following kind: pick an edge e of the tree, and then proceed recursively so in the two connected components of the graph resulting from the removal of e.

These two extremes substantiate our informal explanation of the μ-syntax as a mix of partial composition and simultaneous composition styles.

2.4.3 The interpretation of the μ-syntax in an arbitrary cyclic operad

We next consider the semantic aspect of the μ-syntax relative to unrooted trees that we intuitively brought up in §2.4.1 and §2.4.2, by defining an interpretation of the μ-syntax in an arbitrary cyclic operad characterised as in Section 2.2. We ascribe meaning to the μ-syntax by first translating it to the combinator syntax.

The translation function

$$[[-]] : \mu\texttt{Exp}_{\underline{\mathcal{C}}} \rightarrow \texttt{cTerm}_{\underline{\mathcal{C}}}$$

is defined recursively as follows, wherein the assignment of a combinator to a term $t \in \mu\texttt{Term}_{\underline{\mathcal{C}}}$ is indexed by a variable that is fresh relative to all the variables which appear in t:

- $[[x]]_y = id_{x,y}$,
- if, for each $x \in X$, $[[t_x]]_{\overline{x}}$ is a translation of the term t_x, then

  $$[[\underline{a}\{t_x \,|\, x \in X\}]] = a(\varphi),$$

  where $a(\varphi)$ denotes the combinator corresponding to the simultaneous composition determined by $a \in \underline{\mathcal{C}}(X)$ and $\varphi : x \mapsto ([[t_x]]_{\overline{x}}, \overline{x})$ (see (2.1.1)),
- $[[\mu x.c]]_y = [[c[y/x]\,]]$, and
- $[[\langle s \,|\, t\rangle]] = [[s]]_x \;{}_x{\circ}_y\; [[t]]_y$.

In order to show that $[[-]]$ preserves the equalities from Figure 2.4, we introduce the following notational conventions. For a command $c : X$ (resp. term $X \,|\, t$) and a bijection $\sigma : X' \rightarrow X$, we define

$$c^{\sigma} := c[\ldots, \sigma^{-1}(x)/x, \ldots] \;\; (\text{resp. } \; t^{\sigma} := t[\ldots, \sigma^{-1}(x)/x, \ldots])$$

as a simultaneous substitution (i.e. renaming) of the variables from the set X (guided by σ). One of the basic properties of the introduced substitution is the equality

$$(\mu a.c)^{\sigma} = \mu a.c^{\sigma_a}$$

(for the definition of σ_a, see the paragraph Notation and conventions in the Introduction).

The way c^{σ} is defined indicates that its translation should be the combinator $[[c]]^{\sigma} : X'$. The following lemma ensures that this is exactly the case. In its statement, $[[-]]_X$ denotes the restriction of $[[-]]$ on $\mu\texttt{Exp}_{\underline{\mathcal{C}}}(X)$. Furthermore, for a bijection $\sigma : X' \rightarrow X$, $(-)^{\sigma} : \texttt{cTerm}_{\underline{\mathcal{C}}}(X) \rightarrow \texttt{cTerm}_{\underline{\mathcal{C}}}(\texttt{X}')$ will be the mapping of combinators canonically induced by $\underline{\mathcal{C}}(\sigma) : \underline{\mathcal{C}}(X) \rightarrow \underline{\mathcal{C}}(X')$.

Lemma 2.30. *For a bijection $\sigma : X' \rightarrow X$, $t \in \mu\texttt{Term}_{\underline{\mathcal{C}}}(X)$ and $c \in \mu\texttt{Comm}_{\underline{\mathcal{C}}}(X)$, the following two equalities hold:*

$$[[t^{\sigma}]]_y = [[t]]_y^{\sigma_y} \qquad \textit{and} \qquad [[c^{\sigma}]] = [[c]]^{\sigma}.$$

Proof. By structural induction on t and c. ∎

In order to verify that $[[-]]$ is sound, we shall also need the following result.

Lemma 2.31 (Substitution lemma). *Let $X \cap Y = \emptyset$, $t \in \mu\mathtt{Term}_{\underline{\mathcal{C}}}(Y)$ and $x \in X$. Then, for $s \in \mu\mathtt{Term}_{\underline{\mathcal{C}}}(X)$ and $c \in \mu\mathtt{Comm}_{\underline{\mathcal{C}}}(X)$, the following two equalities hold:*

$$[[s[t/x]]]_u = [[s]]_u \,{}_x\circ_v [[t]]_v \qquad \textit{and} \qquad [[c[t/x]]] = [[c]] \,{}_x\circ_v [[t]]_v.$$

Proof. By structural induction on t.

If t is a variable, say y, then, by (U2) and (EQ), we get

$$[[s[y/x]]]_u = [[s^{id_X^{y/x}}]]_u = [[s]]_u{}^{id_X^{y/x}} = [[s]]_u \,{}_x\circ_v id_{v,y} = [[s]]_u \,{}_x\circ_v [[y]]_v\,,$$

and, analogously,

$$[[c[y/x]]] = [[c^{id_X^{y/x}}]] = [[c]]^{id_X^{y/x}} = [[c]] \,{}_x\circ_z id_{z,y} = [[c]] \,{}_x\circ_z [[y]]_z.$$

If $t = \mu y.c_1$, we proceed by induction on the structure of s, i.e. c.

- If $s = x$, then, again by (U2) and (EQ), we get

$$\begin{aligned}[[x[\mu y.c_1/x]]]_u &= [[\mu y.c_1]]_u = [[c_1[u/y]]] = [[c_1^{id_Y^{u/y}}]] \\ &= [[c_1]]^{id_Y^{u/y}} = [[c_1]] \,{}_y\circ_x id_{x,u} = [[c_1]] \,{}_y\circ_x [[x]]_u = [[c_1[u/y]]] \,{}_u\circ_x [[x]]_u.\end{aligned}$$

- Next, assume that $c : X \cup \{z\}$ satisfies the equality and let $s = \mu z.c$. Denote $U = X\backslash\{x\} \cup \{z\} \cup Y$. By (U2) and (EQ), we have

$$\begin{aligned}[[\mu z.c[\mu y.c_1/x]]]_u &= [[\mu z.(c[\mu y.c_1/x])]]_u = [[c[\mu y.c_1/x][u/z]]] = [[c[\mu y.c_1/x]^{id_U^{u/z}}]] \\ &= [[c[\mu y.c_1/x]]]^{id_U^{u/z}} = ([[c]] \,{}_x\circ_y [[c_1]])^{id_U^{u/z}} = [[c]]^{id_U^{u/z}} \,{}_x\circ_y [[c_1]] \\ &= [[c[u/z]]] \,{}_x\circ_v [[c_1[v/y]]] = [[\mu v.c]]_u \,{}_x\circ_v [[c_1[v/y]]].\end{aligned}$$

- Let $X = X_1 \cup X_2$ and suppose that $c = \langle t_1 \,|\, t_2\rangle$, where $X_1 \,|\, t_1$ and $X_2 \,|\, t_2$ satisfy the claim. Without loss of generality, we can assume that $x \in X_2$. By (A1), we have

$$\begin{aligned}[[\langle t_1 \,|\, t_2\rangle[\mu y.c_1/x]]] &= [[\langle t_1 \,|\, t_2[\mu y.c_1/x]\rangle]] = [[t_1]]_u \,{}_u\circ_v [[t_2[\mu y.c_1/x]]]_v \\ &= [[t_1]]_u \,{}_u\circ_v ([[t_2]]_v \,{}_x\circ_w [[\mu y.c_1]]_w) = ([[t_1]]_u \,{}_u\circ_u [[t_2]]_u) \,{}_x\circ_w [[\mu y.c_1]]_w \\ &= [[\langle t_1 \,|\, t_2\rangle]] \,{}_x\circ_v [[\mu y.c_1]]_v.\end{aligned}$$

- Finally, let $X = \bigcup_{z\in Z} Y_z$ and suppose that $c = \underline{a}\{t_z \,|\, z \in Z\}$, where for all $z \in Z$, $Y_z \,|\, t_z$ satisfy the claim. Suppose, moreover, that for $u \in Z$, $x \in Y_u$. Then, on one hand, we have

$$[[\underline{a}\{t_z \,|\, z \in Z\}[\mu y.c_1/x]]] = [[\underline{a}\{\{t_z \,|\, z \in Z\backslash\{u\}\} \cup \{t_u[\mu y.c_1/x]\}\}]] = a(\varphi),$$

where $\varphi : z \mapsto ([[t_z]]_{\overline{z}}, \overline{z})$, for all $z \in Z\backslash\{u\}$, and $\varphi : u \mapsto ([[t_u[\mu y.c_1/x]]]_{\overline{u}}, \overline{u})$. On the other hand,

$$[[\underline{a}\{t_z \,|\, z \in Z\}]] \,{}_x\circ_v [[\mu y.c_1]]_v = a(\psi_1) \,{}_x\circ_v [[\mu y.c_1]]_v,$$

where $\psi_1 : z \mapsto ([[t_z]]_{\overline{z}}, \overline{z})$, for all $z \in Z$. By Lemma 2.1,

$$a(\psi_1) \,{}_x\circ_v [[\mu y.c_1]]_v = a(\psi_2),$$

where $\psi_2 = \psi_1$ on $Z\backslash\{a\}$, and $\psi_2 : u \mapsto ([[t_u]]_{\overline{u}} \,{}_x\circ_v [[\mu y.c_1]]_v, \overline{u})$. Hence, we need to prove that

$$[[t_u[\mu y.c_1/x]]]_{\overline{u}} = [[t_u]]_{\overline{u}} \,{}_x\circ_v [[\mu y.c_1]]_v,$$

but this equality is exactly the induction hypothesis for the term t_u.

■

Let $=_\mu$ (resp. $=$) be the smallest equivalence relation on $\mu\mathtt{Exp}_{\underline{\mathcal{C}}}$ (resp. $\mathtt{cTerm}_{\underline{\mathcal{C}}}$) generated by the equations of μ-syntax (resp. by the equations of Definition 1.4).

Theorem 2.32. *The translation function* $[[-]] : \mu\mathtt{Exp}_{\underline{\mathcal{C}}} \to \mathtt{cTerm}_{\underline{\mathcal{C}}}$ *is well-defined, i.e., it induces a map from* $\mu\mathtt{Exp}_{\underline{\mathcal{C}}}/_{=_\mu}$ *to* $\mathtt{cTerm}_{\underline{\mathcal{C}}}/_{=}$. *Moreover, the induced map is a bijection.*

Proof. The equation `(MU1)` is valid in the world of combinators, as it gets translated to `(CO)`. As for `(MU2)`, for a command $c : X$, by Lemma 2.31, we get:

$$[[\langle \mu x.c \,|\, t\rangle]] = [[\mu x.c]]_u \; {}_u{\circ}_v \; [[t]]_v = [[c[u/x]]] \; {}_u{\circ}_v \; [[t]]_v = [[c]]^{id_X^{u/x}} \; {}_u{\circ}_v \; [[t]]_v = [[c]] \; {}_x{\circ}_v \; [[t]]_v = [[c[t/x]]].$$

For `(MU3)` and `(MU4)`, we have

$$[[\mu x.c]]_u = [[c[u/x]]] = [[c[y/x][u/y]]] = [[\mu y.c[y/x]]]_u$$

and

$$[[\underline{a}^\sigma\{t_{\sigma(y)} \,|\, y \in Y\}]] = a^\sigma(\varphi') = a^\sigma(\varphi \circ \sigma) = a(\varphi) = [[\underline{a}\{t_x \,|\, x \in X\}]],$$

where $\varphi' : y \mapsto ([[t_{\sigma(y)}]]_{\overline{\sigma(y)}}, \overline{\sigma(y)})$ and $\varphi : \sigma(y) \mapsto ([[t_{\sigma(y)}]]_{\overline{\sigma(y)}}, \overline{\sigma(y)})$, respectively.

The inverse translation is obtained via the correspondence $(-)_x{\circ}_y(-) \;\; \mapsto \;\; \langle \mu x.\_ \,|\, \mu y.\_\rangle$. ■

We define the interpretation of the μ-syntax in an arbitrary cyclic operad $\mathcal{C}$ as the composition

$$[\,[[-]]\,]_{\mathcal{C}} : \mu\mathtt{Exp}_{\underline{\mathcal{C}}} \to \mathcal{C}, \tag{2.4.1}$$

where the interpretation $[-]_{\mathcal{C}} : \mathtt{cTerm}_{\underline{\mathcal{C}}} \to \mathcal{C}$ arises as explained in Section 2.2.

2.4.4 μ-syntax does the job!

The theorem below puts the μ-syntax in line with already established frameworks for defining a cyclic operad.

Theorem 2.33. *The correspondence* $\Phi_X : \mu\mathtt{Comm}_{\underline{\mathcal{C}}}(X)_{/=_\mu} \to \mathtt{T}_{\underline{\mathcal{C}}}(X)$, *canonically induced from the interpretation*

$$[\,[[-]]\,]_{\mathtt{T}_{\underline{\mathcal{C}}}} : \mu\mathtt{Exp}_{\underline{\mathcal{C}}} \to \mathtt{T}_{\underline{\mathcal{C}}},$$

of the μ*-syntax in the free cyclic operad* $\mathtt{T}_{\underline{\mathcal{C}}}$ *(defined in §2.3.3), is a bijection.*

The proof of Theorem 2.33 goes through a new equality $='$ on $\mu\mathtt{Comm}^{nf}_{\underline{\mathcal{C}}}(X)$, as well as suitably tailored decompositions of unrooted trees, necessary for establishing the injectivity of Φ_X. We first describe these decompositions and the equality $='$ and then prove the theorem.

"Pruning" of unrooted trees

We describe an algorithm that takes an ordinary unrooted tree $\mathcal{T}$, a corolla $a \in Cor(\mathcal{T})$ and a variable $v \in FV(a)\backslash FV(\mathcal{T})$, and returns a proper subtree $\mathcal{T}_v$ of $\mathcal{T}$, the subtree "plucked" from a at the junction of v and $\sigma(v)$, where σ is the involution of $\mathcal{T}$. In the sequel, for an arbitrary corolla $b \in Cor(\mathcal{T})$ and $w \in FV(b)\backslash FV(\mathcal{T})$, $S_w(b)$ will denote the corolla adjacent to b along the edge $(w, \sigma(w))$ (if such a corolla exists).

We first specify how to generate the set $Cor(\mathcal{T}_v)^+$ of pairs of a corolla of $\mathcal{T}_v$ and one of its free variables, by the following formal rules:

$$\frac{}{(S_v(a), \sigma(v)) \in Cor(\mathcal{T}_v)^+} \qquad \frac{(b,u) \in Cor(\mathcal{T}_v)^+ \quad w \in FV(b)\backslash(FV(\mathcal{T}) \cup \{u\})}{(S_w(b), \sigma(x)) \in Cor(\mathcal{T}_v)^+}$$

This formal system has the following properties.

Remark 2.34. *Each element* $(S_w(b), \sigma(w)) \in Cor(\mathcal{T}_v)^+$ *is such that* $S_w(b)$ *is adjacent to* b *in* $\mathcal{T}$. *For each* $(b, u) \in Cor(\mathcal{T}_v)^+$, *we have* $b \neq a$. *Intuitively, by iterative application of the second rule, a way to traverse a branch of the tree* $\mathfrak{T}$ *is determined.*

We obtain the set of corollas of $\mathcal{T}_v$ by erasing from the elements of $Cor(\mathcal{T}_v)^+$ the data about the distinguished free variables, i.e. we define

$$Cor(\mathcal{T}_v) = \{b \,|\, (b, u) \in Cor(\mathcal{T}_v)^+ \text{ for some } u \in FV(b)\}.$$

The involution $\sigma_{\mathcal{T}_v}$ of $\mathcal{T}_v$ is defined as

$$\sigma_{\mathcal{T}_v}(z) = \begin{cases} \sigma(z) & \text{if } z \in \left(\bigcup_{b \in Cor(\mathcal{T}_v)} FV(b)\right) \backslash \sigma(v) \\ z & \text{if } z = \sigma(v)\,. \end{cases}$$

We shall denote the algorithm with $\mathcal{P}$, and the result $\mathcal{P}(\mathcal{T}, a, v)$ of instatiating $\mathcal{P}$ on a tree $\mathcal{T}$, a corolla $a \in Cor(\mathcal{T})$, and a variable $v \in FV(a)\backslash FV(\mathcal{T})$ will often be denoted as $\mathcal{T}_v$, as we have just done above. The following claim guarantees that $\mathcal{P}$ is correct.

Lemma 2.35. *For an unrooted tree* $\mathcal{T}$, $a \in Cor(\mathcal{T})$ *and* $v \in FV(a)\backslash FV(\mathcal{T})$, $\mathcal{T}_v$ *is a proper subtree of* $\mathcal{T}$.

Proof. By the construction, we have that $Cor(\mathcal{T}_v) \subseteq Cor(\mathcal{T})$ and that $\mathcal{T}_v$ is connected. By Remark 2.34, it follows that $Cor(\mathcal{T}_v)$ is a proper subset of $Cor(\mathcal{T})$. Finally, since $\sigma_{\mathcal{T}_v} = \sigma$ on $V(\mathcal{T}_v)\backslash FV(\mathcal{T}_v)$, we can conclude that $\mathcal{T}_v$ is indeed a subtree of $\mathcal{T}$. ∎

Lemma 2.36. *For an unrooted tree* $\mathcal{T}$ *and* $a \in Cor(\mathcal{T})$, *the set of unrooted trees* $\mathcal{P}(\mathcal{T}, a)$, *defined by*

$$\mathcal{P}(\mathcal{T}, a) = \{\{a(y_1, \dots, y_n); id_Y\}\} \cup \{\mathcal{T}_v \,|\, v \in Y\backslash FV(\mathcal{T})\},$$

where $Y = \{y_1, \dots, y_n\}$, *is a decomposition of* $\mathcal{T}$.

Proof. The proof goes by induction on the cardinality of $Y\backslash FV(\mathcal{T})$. ∎

From the point of view of the appropriate composition of classes of trees, Lemma 2.36 gives us the following result.

Corollary 2.37. *Let* $\mathcal{T}$ *be an unrooted tree and let* $a \in Cor(\mathcal{T})$. *Suppose that* $FV(\mathcal{T}) = X$ *and* $FV(a) = Y$, *where* $Y = \{y_1, \dots, y_n\}$. *Let* $I = \{i_1, \dots i_k\} = \{i \in \{1, \dots, n\} \,|\, y_i \in FV(a)\backslash X\}$. *Then, if* $\mathcal{P}(\mathcal{T}, a) = \{\{a(y_1, \dots, y_n); id_Y\}\} \cup \{\mathcal{T}_{y_i} \,|\, i \in I\}$, *we have that*

$$[\mathcal{T}]_\alpha = ((([\{a(y_1, \dots, y_n); id_Y\}]_\alpha \; {}_{y_{i_1}}\!\bullet_{\sigma(y_{i_1})} [\mathcal{T}_{y_{i_1}}]_\alpha) \cdots) \; {}_{y_{i_k}}\!\bullet_{\sigma(y_{i_k})} [\mathcal{T}_{y_{i_k}}]_\alpha\,.$$

Proof. Given that the trees from $\mathcal{P}(\mathcal{T}, a)$ have mutually disjoint sets of corollas and variables, the equality holds since the composition on the right-hand can be "calculated" without any "precautionary" renaming. ∎

Lemma 2.38. *If an unrooted tree* $\mathcal{T}$ *has at least two corollas, then there exists* $c \in Cor(\mathcal{T})$, *such that* $FV(c)\backslash FV(\mathcal{T})$ *is a singleton.*

Proof. Suppose that $FV(\mathcal{T}) = X$ and let σ be the involution of $\mathcal{T}$. We proceed by induction on the number n of corollas of $\mathcal{T}$.

For the base case, suppose that $Cor(\mathcal{T}) = \{a, b\}$. Then there exist $x \in FV(a)$ and $y \in FV(b)$ such that $\sigma(x) = y$, while all other variables of $\mathcal{T}$ are fixpoints of σ. Hence, $FV(a)\backslash FV(\mathcal{T}) = \{x\}$ and $FV(b)\backslash FV(\mathcal{T}) = \{y\}$, i.e. a and b both satisfy the claim.

Assume now that $\mathcal{T}$ has n corollas, where $n > 2$. Let $a \in Cor(\mathcal{T})$, $FV(a) = Y = \{y_1, \dots, y_n\}$, be such that there exists $v \in Y\backslash X$. If v is the unique such variable, we are done. If not, consider the $\mathcal{P}(\mathcal{T}, a) = \{\{a(y_1, \ldots, y_n); id_Y\}\} \cup \{\mathcal{T}_u \,|\, u \in Y\backslash X\}$ of $\mathcal{T}$. If $Cor(\mathcal{T}_v) = \{S_v(a)\}$, by the definition of $\mathcal{P}$, we know that $FV(S_v(a))\backslash X = \{\sigma(v)\}$. Therefore, since $Cor(\mathcal{T}_v) \subseteq Cor(\mathcal{T})$, $S_v(a)$ is a corolla that satisfies the claim. If $\mathcal{T}_v$ contains more than one corolla, by the induction hypothesis on $\mathcal{T}_v$, we get $b \in Cor(\mathcal{T}_v)$ such that $FV(b)\backslash FV(\mathcal{T}_v) = \{w\}$. Since $FV(b)\backslash X \subseteq FV(b)\backslash FV(\mathcal{T}_v)$, we know that either $FV(b)\backslash X = \{w\}$, or $FV(b)\backslash X = \emptyset$. The latter is impossible because b would be the only corolla of $\mathcal{T}$. ∎

Let $\mathcal{T}$ and c be as in Lemma 2.38 and let $FV(c)\backslash FV(\mathcal{T}) = \{v\}$. We shall denote with $\mathcal{T}_{/c}$ the unrooted tree determined by $Cor(\mathcal{T}_{/c}) = Cor(\mathcal{T})\backslash\{c\}$ and its involution $\sigma_{/c}$, which agrees with the involution σ of $\mathcal{T}$ everywhere, except on $\sigma(v)$, which is a fixpoint of $\sigma_{/c}$. Lemma 2.38 guarantees that $\mathcal{T}_{/c}$ is well-defined.

We now establish a non-inductive characterisation of the output of the algorithm $\mathcal{P}$.

Lemma 2.39. *Let $\mathcal{T}$ be an unrooted tree with involution σ and let $a \in Cor(\mathcal{T})$ and $v \in FV(a)\backslash FV(\mathcal{T})$. The following properties are equivalent for a subtree $\mathcal{T}'$ of $\mathcal{T}$:*

1. *$\mathcal{T}' = \mathcal{P}(\mathcal{T}, a, v)$,*
2. *$\sigma(v) \in FV(\mathcal{T}')$ and $FV(\mathcal{T}')\backslash\{\sigma(v)\} \subseteq FV(\mathcal{T})$.*

Proof. That (1) implies (2) is clear.

We prove that (2) implies (1) by induction on the number n of corollas of $\mathcal{T}'$.

If $n = 1$, then, as $\sigma(v) \in FV(\mathcal{T}')$, we have that $S_v(a)$ is the only corolla of $\mathcal{T}'$, and the conclusion follows since, by the assumption, $FV(\mathcal{T}')\backslash\{\sigma(v)\} = FV(S_v(a))\backslash\{\sigma(v)\} \subseteq X$, i.e. $FV(S_v(a))\backslash\{X \cup \{\sigma(v)\}\} = \emptyset$.

Suppose that $n \geq 2$, and let, by Lemma 2.38, $c \in Cor(\mathcal{T}')$ be such that $FV(c)\backslash FV(\mathcal{T}')$ is a singleton, say $\{y_i\}$, wherein $FV(c) = Y = \{y_1, \ldots, y_n\}$. If $c = S_v(a)$, then it follows easily that $\mathcal{T}' = \mathcal{T}_v$. If not, by applying the induction hypothesis on $\mathcal{T}'_{/c'}$, we get that $\mathcal{T}'_{/c} = \mathcal{P}(\mathcal{T}_{/c}, a, v)$. Observe that $(S_{y_i}(c), w) \in Cor(\mathcal{T}'_{/c})^+$, for some $w \in FV(S_{y_i}(c))$ different from $\sigma(u)$. By instantiating $\mathcal{P}$ on $(S_{y_i}(c), w)$ and $\sigma(u)$, we get the pair (c, y_i), and the claim follows because $Y\backslash(FV(\mathcal{T}) \cup \{y_i\}) = \emptyset$ (i.e. the algorithm stops) and because $\mathcal{T}'_{/c} \cup \{c(y_1, \ldots, y_n); id_Y\}$ is a decomposition of $\mathcal{T}'$. ∎

For the following two lemmas, recall the definition of the simultaneous composition (2.1.1) for entries-only cyclic operads. We shall instantiate it on the cyclic operad of classes of unrooted trees, described in §2.3.3.

Lemma 2.40. *Let $a \in \underline{\mathcal{C}}(X)$, where $X = \{x_1, \ldots, x_n\}$, and let, for all $x_i \in X$, $\gamma : x_i \mapsto ([\mathcal{T}_{x_i}]_\alpha, \overline{x_i})$ be an assignment for which the simultaneous composition $[\{a(x_1, \ldots, x_n); id_X\}]_\alpha(\gamma)$ is well-defined. Then the following properties hold.*

a) The α-equivalence class $[\{a(x_1, \ldots, x_n); id_X\}]_\alpha(\gamma)$ admits a representative $\mathfrak{T}$, such that $a \in Cor(\mathfrak{T})$.

b) If $\mathfrak{T}$ is a representative of $[\{a(x_1, \ldots, x_n); id_X\}]_\alpha(\gamma)$, such that $a \in Cor(\mathfrak{T})$, and if σ is the involution of $\mathfrak{T}$, then each class $[\mathfrak{T}_{x_i}]_\alpha$ admits the unrooted tree $\mathcal{P}(\mathfrak{T}, a, x_i)^{\rho_i}$, where ρ_i renames $\sigma(x_i)$ to $\overline{x_i}$, as a representative.

Proof. Observe that there are two stages of renaming involved in forming the simultaneous composition $[\{a(x_1, \ldots, x_n); id_X\}]_\alpha(\gamma)$. By (2.1.1), we first rename the free variables of the corolla a, obtaining in this way the composition

$$(\cdots([\{a^\sigma(x'_1, \ldots, x'_n); id_{X'}\}]_\alpha \, {}_{x'_1}\bullet_{\overline{x_1}} \, [\mathfrak{T}_{x_1}]_\alpha) \cdots) \, {}_{x'_n}\bullet_{\overline{x_n}} \, [\mathfrak{T}_{x_n}]_\alpha,$$

where $X' = \{x'_1, \ldots, x'_n\}$ and $\sigma : X' \to X$ is defined by $\sigma(x'_i) = x_i$, which is then "calculated" by the definition of ${}_x\bullet_y$ from §2.3.3. This calculation involves the renaming of variables of all the trees from the above composition, in such a way that the resulting trees have mutually disjoint sets of variables, i.e. it goes though the simultaneous composition

$$(\cdots([\{a^{\sigma\circ\tau}(y_1,\ldots,y_n); id_Y\}]_{\alpha\ y'_1}\bullet_{\overline{y_1}}\ [\mathcal{T}_{x_1}^{\tau_1\cup id_{FV(\mathcal{T}_{x_1})\backslash\{\overline{x_1}\}}}]_\alpha)\cdots)\ {}_{y'_n}\bullet_{\overline{y_n}}\ [\mathcal{T}_{x_n}^{\tau_n\cup id_{FV(\mathcal{T}_{x_n})\backslash\{\overline{x_n}\}}}]_\alpha,$$

where $Y = \{y_1, \ldots, y_n\}$, $\tau : Y \to X'$ is defined by $\tau(y_i) = x'_i$ and each $\tau_i : V_i \to (V(\mathcal{T}_{x_i})\backslash FV(\mathcal{T}_{x_i})) \cup \{\overline{x_i}\}$ is such that $\tau_i(\overline{y_i}) = \overline{x_i}$. The resulting class now has as a representative the tree $\mathcal{T}'$, such that

$$Cor(\mathcal{T}') = \{a^{\sigma\circ\tau}(y_1,\ldots,y_n)\} \cup \bigcup_{1\leq i\leq n} Cor(\mathcal{T}_{x_i}^{\tau_i})$$

and whose involution σ' is defined in the obvious way.

The first claim holds, since, thanks to the equivariance axiom (EQ) for ${}_x\bullet_y$, we can turn $\mathcal{T}'$ into an unrooted tree $\mathcal{T}$ that has a as a corolla, by "undoing" the renaming $\sigma\circ\tau$. Clearly, if some variable x_i appears in $\mathcal{T}'$, but did not originally come from the corolla a, this variable has to be renamed too, in order to ensure that all the variables of $\mathcal{T}$ are distinct. Therefore,

$$Cor(\mathcal{T}) = \{a(x_1,\ldots,x_n)\} \cup \bigcup_{1\leq i\leq n} Cor((\mathcal{T}_{x_i}^{\tau_i})^{\kappa_i\cup id_{FV(\mathcal{T}_{x_i})\backslash\{\overline{x_i}\}}}),$$

where $\kappa_i : U_i \cup \{\overline{z_i}\} \to V_i \cup \{\overline{y_i}\}$ is such that $\kappa_i(\overline{z_i}) = \overline{y_i}$ and the distinctness requirement for the variables of $\mathcal{T}$ is satisfied. The involution σ of $\mathcal{T}$ is defined from σ' in the obvious way.

For the second claim, fix an $i \in \{1, \ldots, n\}$. Observe that we have that

$$(\mathcal{T}_{x_i}^{\tau_i})_{x_i}^{\nu_i} =_\alpha \mathcal{T}_{x_i},$$

where ν_i renames $\overline{y_i}$ to $\overline{x_i}$. Also, we have that

$$\mathcal{T}_{x_i}^{\tau_i} =_\alpha ((\mathcal{T}_{x_i}^{\tau_i})^{\kappa_i\cup id_{FV(\mathcal{T}_{x_i})\backslash\{\overline{x_i}\}}})^{\pi_i},$$

where π_i renames $\overline{z_i}$ to $\overline{y_i}$. Therefore,

$$(((\mathcal{T}_{x_i}^{\tau_i})^{\kappa_i\cup id_{FV(\mathcal{T}_{x_i})\backslash\{\overline{x_i}\}}})^{\pi_i})^{\nu_i} =_\alpha \mathcal{T}_{x_i},$$

i.e. each class $[\mathcal{T}_{x_i}]_\alpha$ admits as a representative $((\mathcal{T}_{x_i}^{\tau_i})^{\kappa_i\cup id_{FV(\mathcal{T}_{x_i})\backslash\{\overline{x_i}\}}})^{\rho_i}$, where ρ_i renames $\overline{z_i} = \sigma(x_i)$ to $\overline{x_i}$. Observe that $(\mathcal{T}_{x_i}^{\tau_i})^{\kappa_i\cup id_{FV(\mathcal{T}_{x_i})\backslash\{\overline{x_i}\}}}$ is a subtree of $\mathcal{T}$. That we indeed have that

$$(\mathcal{T}_{x_i}^{\tau_i})^{\kappa_i\cup id_{FV(\mathcal{T}_{x_i})\backslash\{\overline{x_i}\}}} = \mathcal{P}(\mathcal{T}, a, x_i)$$

is clear by considering the non-inductive criterion from Lemma 2.39: $\sigma(x_i)$ is a free variable of the tree on the left-hand side, and all the remaining free variables of that tree are free variables of $\mathcal{T}$. ■

Lemma 2.41. *Let $a \in \underline{\mathcal{C}}(X)$, where $X = \{x_1, \ldots, x_n\}$, and let, for all $x_i \in X$, $\gamma : x_i \mapsto ([\mathcal{T}_{x_i}]_\alpha, \overline{x_i})$ and $\tau : x_i \mapsto ([\mathcal{T}'_{x_i}]_\alpha, \tilde{x}_i)$ be assignments for which the simultaneous compositions*

$$[\{a(x_1,\ldots,x_n); id_X\}]_\alpha(\gamma) \quad \textit{and} \quad [\{a(x_1,\ldots,x_n); id_X\}]_\alpha(\tau)$$

are well-defined. Then, if $[\{a(x_1,\ldots,x_n); id_X\}]_\alpha(\gamma) = [\{a(x_1,\ldots,x_n); id_X\}]_\alpha(\tau)$, we have that $[\mathcal{T}_{x_i}]_\alpha^\kappa = [\mathcal{T}'_{x_i}]_\alpha$ for all $x_i \in X$, where κ renames $\overline{x_i}$ to $\tilde{x}_i$.

Proof. By Lemma 2.40(a), for

$$[\{a(x_1,\ldots,x_n); id_X\}]_\alpha(\gamma) = [\{a(x_1,\ldots,x_n); id_X\}]_\alpha(\tau) = [\mathcal{T}]_\alpha,$$

we can assume that the representative $\mathcal{T}$ is such that it has $a \in Cor(\mathcal{T})$. Let σ be the involution of $\mathcal{T}$. By applying twice Lemma 2.40(b), we get that

$$[\mathcal{T}_{x_i}]^\kappa_\alpha = [\mathcal{P}(\mathcal{T},a,x_i)^{\rho_i}]^\kappa_\alpha = [\mathcal{T}'_{x_i}]_\alpha,$$

where ρ_i renames $\sigma(x_i)$ to $\overline{x_i}$, which proves the claim. ∎

The equivalence relation $='$ on $\mu\mathtt{Comm}^{nf}_{\underline{\mathcal{C}}}$

Let $a \in \underline{\mathcal{C}}(X)$ and let $\sigma : x \mapsto t_x$ be an association of terms to variables from X, such that the command $\underline{a}\{\sigma\}$ is well-typed. The equivalence relation $='$ is the smallest equivalence relation generated by equalities

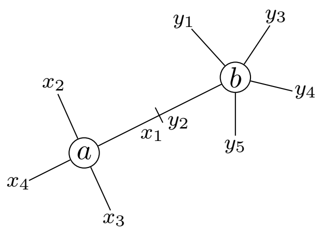
$$\boxed{\underline{a}\{\sigma\} =' c[\mu x.\underline{a}\{\sigma[x/x]\}/y]}$$

where $\sigma(x) = \mu y.c$ and $\sigma[x/x]$ denotes the same association as σ, except for x, to which it associates x itself. We, moreover, assume that $='$ is congruent with respect to (MU3), (MU4) and substitution.

Remark 2.42. *Observe that, if $\underline{a}\{\sigma\} =' c[\mu x.\underline{a}\{\sigma[x/x]\}/y]$, and if $\underline{a}\{\sigma\}$ is a normal form, then this is also true for the command $c[\mu x.\underline{a}\{\sigma[x/x]\}/y]$. Therefore, $='$ is well-defined on $\mu\mathtt{Comm}^{nf}_{\underline{\mathcal{C}}}$.*

The intuition behind these equalities is again about equating commands that reflect two ways to build the same unrooted tree.

Example 2.43. Consider the unrooted tree $\mathcal{T} = \{a(x_1,x_2,x_3,x_4), b(y_1,y_2,y_3,y_4,y_5);\sigma\}$, where $\sigma = (x_1\, y_2)$, represented pictorially as



The commands equated by $='$ reflect the two possible ways to build $\mathcal{T}$ by means of simultaneous grafting: we could pick either the corolla a and graft to it the surrounding trees, or we can do the same by choosing first the corolla b. In the language of the μ-syntax, the two constructions are described by the left hand side and the right hand side of the equality

$$\underline{a}\{\mu y_2.\underline{b}\{y_1,y_2,y_3,y_4,y_5\},x_2,x_3,x_4\} =' \underline{b}\{y_1,\mu x_1.\underline{a}\{x_1,x_2,x_3,x_4\},y_3,y_4,y_5\},$$

respectively. Observe that, from the tree-wise perspective, $='$ enables us to "move between two adjacent corollas", i.e. it enables us to "move along a path in a tree". As we shall see, this feature will be crucial for the proof of injectivity of Theorem 2.33. □

The proof of the following lemma shows that $='$ is a "macro" derivable from $=_\mu$.

Lemma 2.44. *For any $c_1, c_2 \in \mu\mathtt{Comm}^{nf}_{\underline{\mathcal{C}}}$, if $c_1 =' c_2$, then $c_1 =_\mu c_2$.*

Proof. If $\underline{a}\{\sigma\} =' c[\mu x.\underline{a}\{\sigma[x/x]\}/y]$, then $\sigma(x) = \mu y.c$, which justifies the following sequence of equalities:

$$\underline{a}\{\sigma\} =_\mu \langle \mu x.\underline{a}\{\sigma[x/x]\} \,|\, \mu y.c\rangle =_\mu \langle \mu y.c \,|\, \mu x.\underline{a}\{\sigma[x/x]\}\rangle =_\mu c[\mu x.\underline{a}\{\sigma[x/x]\}/y].$$ ∎

The equality $='$ (denoted differently) appears in the work [Lam07] of Lamarche, where it is called Adjunction and used in the context of the so-called *reversible terms*. Although the Adjunction rule materialises the same intuition about unrooted trees, there, unlike in this thesis, it is not derived from a more primitive notion of equality.

The proof of Theorem 2.33

We first "unfold" the definition of the interpretation function $[\,[[-]]\,]_{\mathtt{T}_{\underline{\mathcal{C}}}}$, denoted from now on simply with $\underline{\Phi}$:

- $\underline{\Phi}_y(x) = [\{(x,y); id_{\{x,y\}}\}]_\alpha$,
- if, for each $x_i \in \{x_1,\dots,x_n\}$, $\underline{\Phi}_{\overline{x}_i}(t_{x_i}) = [\mathcal{T}_{x_i}]_\alpha$, then
$$\underline{\Phi}(\underline{a}\{t_{x_1},\dots,t_{x_n}\}) = [\{a(x_1,\dots,x_n); id_X\}]_\alpha(\varphi),$$
where $\varphi : x_i \mapsto ([\mathcal{T}_{x_i}]_\alpha, \overline{x}_i)$ (see (2.1.1)),
- $\underline{\Phi}_y(\mu x.c) = (\underline{\Phi}(c))^\kappa$, where κ renames x to y, and
- if $\underline{\Phi}_x(s) = [\mathcal{T}_s]_\alpha$ and $\underline{\Phi}_y(t) = [\mathcal{T}_t]_\alpha$, then $\underline{\Phi}(\langle s \,|\, t\rangle) = [\mathcal{T}_s]_\alpha\, {}_x\bullet_y\, [\mathcal{T}_t]_\alpha$,

where the assignment of an α-equivalence class of unrooted trees to a term $t \in \mu\mathtt{Term}_{\underline{\mathcal{C}}}$ is indexed by a fresh variable y involved in the corresponding interpretation $[[t]]_y$.

By Theorem 2.32, $\underline{\Phi}$ is well-defined. We prove that it is both injective and surjective.

Surjectivity. Suppose given an α-equivalence class $[\mathcal{T}]_\alpha \in \mathtt{T}_{\underline{\mathcal{C}}}(X)$. If $\mathcal{T} = \{(x,y); id_{\{x,y\}}\}$, then $\underline{\Phi}(\langle x \,|\, y\rangle) = [\{(x,y); id_{\{x,y\}}\}]_\alpha$.

Suppose now that $\mathcal{T}$ is an ordinary unrooted tree. We proceed by induction on the number k of corollas of $\mathcal{T}$. Let $a \in Cor(\mathcal{T})$ be such that $FV(a) = Y$, where $Y = \{y_1,\dots,y_n\}$.

If a is the only corolla of $\mathcal{T}$, then $\underline{\Phi}(\underline{a}\{y_1,\dots,y_n\}) = [\{a(y_1,\dots,y_n); id_Y\}]_\alpha$.

Suppose that a is not the only corolla of $\mathcal{T}$, i.e. that $k \geq 2$, and let σ be the involution of $\mathcal{T}$. Let $I = \{i \in \{1,\dots,n\} \,|\, y_i \in FV(a)\backslash X\}$ and $J = \{1,\dots,n\}\backslash I$. By the induction hypothesis for each $\mathcal{P}(\mathcal{T}, a, y_i) = \mathcal{T}_{y_i}$ (recall from §2.4.4 that $\mathcal{P}$ is the "pruning" algorithm), for $i \in I$, we get a set
$$\{c_i \in \mu\mathtt{Comm}_{\underline{\mathcal{C}}} \,|\, i \in I \text{ and } \underline{\Phi}(c_i) = [\mathcal{T}_{y_i}]_\alpha\}.$$
We now set for all $i \in I$, $t_{y_i} = \mu\sigma(y_i).c_i$, and for all $j \in J$, $t_{y_j} = y_j$, and we claim that $\underline{\Phi}(a\{t_{y_1},\dots,t_{y_n}\}) = [\mathcal{T}]_\alpha$. By the definition of $\underline{\Phi}$, we have
$$\underline{\Phi}(a\{t_{y_k} \,|\, k \in \{1,\dots,n\}\}) = [\{a(y_1,\dots,y_n); id_X\}]_\alpha(\varphi),$$
where
$$\varphi : y_k \mapsto \begin{cases} ([\mathcal{T}_{y_i}]_\alpha^{\kappa_i}, z_i) & \text{if } k = i \text{ for some } i \in I \\ ([\{(y_j, \underline{y_j}); id_{\{y_j,\underline{y_j}\}}\}]_\alpha, \underline{x_j}) & \text{if } k = j \text{ for some } j \in J \end{cases}$$
with $[\mathcal{T}_{y_i}]_\alpha^{\kappa_i} = \underline{\Phi}_{z_i}(\mu\sigma(y_i).c_i)$ being the class associated to the term $\mu\sigma(y_i).c_i$ with respect to the interpretation under the fresh variable z_i. Therefore, if $I = \{i_1,\dots,i_{m_I}\}$ and $J = \{j_1,\dots,j_{m_J}\}$, by the axiom `(U1)`, $\underline{\Phi}(a\{t_{y_k} \,|\, k \in \{1,\dots,n\}\})$ is equal to
$$(\cdots([\{a(y_1,\dots,y_n); id_Y\}]_\alpha^{\kappa_{j_1}\kappa_{j_2}\cdots\kappa_{j_{m_J}}}\ {}_{y_{i_1}}\bullet_{z_{i_1}}\ [\mathcal{T}_{y_{i_1}}]_\alpha^{\kappa_{i_1}})\cdots)\ {}_{y_{i_{m_I}}}\bullet_{z_{i_{m_I}}}\ [\mathcal{T}_{y_{i_{m_I}}}]_\alpha^{\kappa_{i_{m_I}}}$$
where each κ_{j_m}, $1 \leq m \leq m_J$ is the renaming of y_{j_k} to y_{j_k}, i.e. the identity on Y, and each κ_{i_m}, $1 \leq m \leq m_I$, is the renaming of z_{i_k} to $\sigma(x_{i_k})$. Finally, by `(EQ)`, we have
$$\underline{\Phi}(a\{t_{y_k} \,|\, k \in \{1,\dots,n\}\}) = (([\{a(y_1,\dots,y_n); id_Y\}]_\alpha\ {}_{y_{i_1}}\bullet_{\sigma(y_{i_1})}\ [\mathcal{T}_{y_{i_1}}]_\alpha)\cdots)\ {}_{y_{i_{m_I}}}\bullet_{\sigma(y_{i_{m_I}})}\ [\mathcal{T}_{y_{i_{m_I}}}]_\alpha,$$

and, consequently, by Corollary 2.37, that $\underline{\Phi}(a\{t_{y_k} \,|\, k \in \{1,\dots,n\}\}) = [\mathcal{T}]_\alpha$.

Injectivity. Notice that, in order to establish the injectivity of $\underline{\Phi}$, it suffices to prove it for commands $c_1, c_2 \in \mu\mathtt{Comm}^{nf}_{\underline{\mathcal{C}}}$. Indeed, since for an arbitrary command c, by Theorem 2.32, we know that $[[c]] = [[nf(c)]]$, and consequently that $\underline{\Phi}(c) = \underline{\Phi}(nf(c))$, then, from the equality

$$\underline{\Phi}(nf(c_1)) = \underline{\Phi}(c_1) = \underline{\Phi}(c_2) = \underline{\Phi}(nf(c_2)),$$

by the injectivity for commands that are normal forms, we can conclude that

$$c_1 =_\mu nf(c_1) =_\mu nf(c_2) =_\mu c_2.$$

By Lemma 2.44, the injectivity for normal forms follows if we show that, if $\underline{\Phi}(c_1) = \underline{\Phi}(c_2)$, then $c_1 =' c_2$. We continue by comparing the head symbols of c_1 and c_2.

If c_1 and c_2 have the same head symbol, we proceed by induction on the structure of c_1 and c_2. Suppose that $a \in \underline{\mathcal{C}}(X)$, where $X = \{x_1,\dots,x_n\}$, and that $c_1 = \underline{a}\{s_1,\dots,s_n\} = \underline{a}\{\sigma\}$ and $c_2 = \underline{a}\{t_1,\dots,t_n\} = \underline{a}\{\sigma'\}$. The assumption $\underline{\Phi}(c_1) = \underline{\Phi}(c_2)$ means that

$$[\{a(x_1,\dots,x_n); id_X\}]_\alpha(\varphi) = [\{a(x_1,\dots,x_n); id_X\}]_\alpha(\psi),$$

where $\varphi : x_i \mapsto (\underline{\Phi}_{\tilde{x}_i}(s_i), \tilde{x}_i)$ and $\psi : x_i \mapsto (\underline{\Phi}_{\overline{x}_i}(t_i), \overline{x}_i)$, and consequently, by Lemma 2.41, that for all $x_i \in X$, $(\underline{\Phi}_{\tilde{x}_i}(s_i))^{\kappa_i} = \underline{\Phi}_{\overline{x}_i}(t_i)$, where κ_i renames $\tilde{x}_i$ to $\overline{x}_i$. The claim holds by the reflexivity of $='$ if all s_i and t_i are variables: if $s_i = u$ and $t_i = v$, then

$$[\{(u,\overline{x}_i); id_{\{u,\overline{x}_i\}}\}]_\alpha = (\underline{\Phi}_{\tilde{x}_i}(u))^{\kappa_i} = \underline{\Phi}_{\overline{x}_i}(v) = [\{(v,\overline{x}_i); id_{\{v,\overline{x}_i\}}\}]_\alpha,$$

and, therefore, it must be the case that $u = v$.

Suppose, therefore, that $s_i = \mu u.c_i$ and $t_i = \mu v.c_i'$. We then have

$$[[c_i^{\tau_1}]] = [[c_i]]^{\tau_1} = [[s_i]]^{\kappa_i}_{\tilde{x}_i} = [[t_i]]_{\overline{x}} = [[c_i']]^{\tau_2} = [[c_i'^{\tau_2}]],$$

and consequently $\underline{\Phi}(c_i^{\tau_1}) = \underline{\Phi}(c_i'^{\tau_2})$, where τ_1 renames u to $\overline{x}_i$ and τ_2 renames v to $\overline{x}_i$. By the induction hypothesis, we now have $c_i^{\tau_1} =' c_i'^{\tau_2}$ and, consequently, we get that

$$\begin{array}{rclcll}\underline{a}\{t_1,\dots,t_n\} & =' & c_i[\mu x_i.\underline{a}\{\sigma[x_i/x_i]\}/u] & = & c_i^{\tau_1}[\mu x_i.\underline{a}\{\sigma[x_i/x_i]\}/\overline{x}_i] & \\ & =' & c_i'^{\tau_2}[\mu x_i.\underline{a}\{\sigma[x_i/x_i]\}/\overline{x}_i] & = & c_i'[\mu x_i.\underline{a}\{\sigma[x_i/x_i]\}/v] & =' \;\; \underline{a}\{s_1,\dots,s_n\}.\end{array}$$

Suppose now that c_1 and c_2 have different head symbols, i.e. that for some $a \in \underline{\mathcal{C}}(X)$ and $b \in \underline{\mathcal{C}}(Y)$, where $X = \{x_1,\dots,x_n\}$ and $Y = \{y_1,\dots,y_m\}$, we have $c_1 = \underline{a}\{s_1,\dots,s_n\} = \underline{a}\{\sigma_1\}$ and $c_2 = \underline{b}\{t_1,\dots,t_m\} = \underline{b}\{\sigma_2\}$, and let $\underline{\Phi}(c_1) = [\mathcal{T}_{c_1}]_\alpha$ and $\underline{\Phi}(c_2) = [\mathcal{T}_{c_2}]_\alpha$. Let $\mathcal{T}$ be a representative of $[\mathcal{T}_{c_1}]_\alpha = [\mathcal{T}_{c_2}]_\alpha$. Observe that two groups of renamings feature in the transitions from c_1 and c_2 to $\mathcal{T}$: the first one contains the renamings specified by the definitions of the simultaneous compositions $\underline{\Phi}(c_1)$ and $\underline{\Phi}(c_2)$, and the second one contains the renamings given by the α-equivalence of $\mathcal{T}_{c_1}$ and $\mathcal{T}$, and $\mathcal{T}_{c_2}$ and $\mathcal{T}$. However, by (MU4), all the renamings of parameters and variables of c_1 and c_2 made in defining $\mathcal{T}$ can be also performed on c_1 and c_2 themselves, leading to commands $c_1' =_\mu c_1$ and $c_2' =_\mu c_2$, such that $\underline{\Phi}(c_1') = \underline{\Phi}(c_2') = [\mathcal{T}]_\alpha$ and such that $\mathcal{T}$ shares the same sets of parameters and variables with both c_1' and c_2'. Hence, we can assume that $\mathcal{T}$ already shares the same sets of parameters and variables with c_1 and c_2. This, in particular, means that $a, b \in Cor(\mathcal{T})$.

Let $x \in X$ be such that $b \in Cor(\mathcal{P}(\mathcal{T}, a, x))$. By the construction of $\mathcal{T}$, the parameter $\underline{b}$ appears in $\sigma_1(x) = \mu u.c$. We define the *distance between* $\underline{a}$ *and* $\underline{b}$ *in* c_1 as the natural number $d_{c_1}(a,b)$ determined as follows.

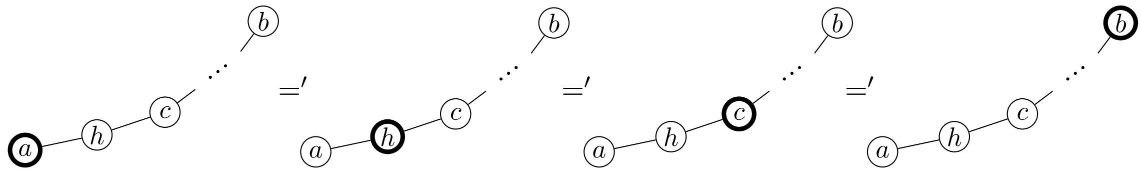
- If $\underline{b}$ is the head symbol of c, then $d_{c_1}(a,b) = 1$.

- If $\underline{h}$ is the head symbol of c, $h \neq b$, then $d_{c_1}(a, b) = d_c(h, b) + 1$.

We prove that $c_1 =' c_2$ by induction on $d_{c_1}(a, b)$. If $d_{c_1}(a, b) = 1$, then, for some $y \in FV(b)$, we have that $\sigma_1(x) = \mu y.\underline{b}\{\sigma_2[y/y]\}$. Therefore,

$$\begin{array}{rcl} \underline{a}\{\sigma_1\} & =' & \underline{b}\{\sigma_2[y/y]\}[\mu x.\underline{a}\{\sigma_1[x/x]\}/y] \\ & = & \underline{b}\{\sigma_2[\mu x.\underline{a}\{\sigma_1[x/x]\}/y]\} \\ & =' & \underline{b}\{\sigma_2\}. \end{array}$$

If $d_{c_1}(a, b) \geq 2$, then, since $d_{c_1}(a, h) = 1$ (where h is as above), we have that $c_1 =' c[\mu x.\underline{a}\{\sigma_1[x/x]\}/u]$. On the other hand, by the induction hypothesis for $d_c(h, b) < n$, we have that $c_2 =' c[\mu x.\underline{a}\{\sigma_1[x/x]\}/u]$, and the conclusion follows by the transitivity of $='$. The iterative application of the equality $='$, implicit in the induction argument, which reduces the distance between $\underline{a}$ and $\underline{b}$, can be illustrated as follows



This completes the proof of Theorem 2.33.

Note that we have in fact *two* bijections: $\mu\texttt{Comm}_{\mathcal{C}}(X)/_{=_\mu} \simeq \mu\texttt{Comm}^{nf}_{\mathcal{C}}(X)/_{='} \simeq \texttt{T}_{\mathcal{C}}(X)$, the first one being induced via normal forms of $\rightsquigarrow$: we have that $nf(c_1) =' nf(c_2)$ implies $c_1 =_\mu c_2$, and conversely, if $c_1 =_\mu c_2$, then $\underline{\Phi}(nf(c_1) = \underline{\Phi}(nf(c_2))$ implies $nf(c_1) =' nf(c_2)$.

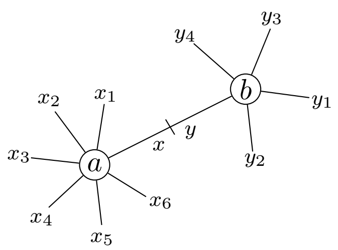
2.5 The equivalence established

We finally show how the μ-syntax, together with the syntactic formalism of unrooted trees suited to it, allows us to prove Theorem 2.22.

Suppose that $(\underline{\mathcal{C}}, \delta)$ is an $\mathcal{M}$-algebra. We build a cyclic operad, as described by Definition 1.4, as follows.

We distinguish the identities, by setting $id_{x,y} = \delta_{\{x,y\}}([\{(x, y); id_{\{x,y\}}\}]_\alpha)$.

The definition of the partial composition operation ${}_x\circ_y$ is derived by considering restrictions of δ to unrooted trees with two corollas:



$$\overset{\alpha}{\longmapsto} \qquad a\,{}_x\circ_y\, b$$

Formally, for $a \in \underline{\mathcal{C}}(X)$ and $b \in \underline{\mathcal{C}}(Y)$, the partial composition operation

$${}_x\circ_y : \underline{\mathcal{C}}(X) \times \underline{\mathcal{C}}(Y) \to \underline{\mathcal{C}}(X\backslash\{x\} \cup Y\backslash\{y\})$$

is characterised via $\delta_{X\backslash\{x\}\cup Y\backslash\{y\}} : \mathcal{M}(\underline{\mathcal{C}})(X\backslash\{x\} \cup Y\backslash\{y\}) \to \underline{\mathcal{C}}(X\backslash\{x\} \cup Y\backslash\{y\})$ as

$$a\,{}_x\circ_y\, b = \delta_{X\backslash\{x\}\cup Y\backslash\{y\}}([\{a(x, \dots); id_X\}]_\alpha\, {}_x\bullet_y\, [\{b(y, \dots); id_Y\}]_\alpha),$$

where ${}_x\bullet_y$ is the operation on (classes of) unrooted trees defined in §2.3.3.

As a structure morphism of $\mathcal{M}$-algebra $(\underline{\mathcal{C}}, \delta)$, δ satisfies the coherence conditions given by commutations of the following two diagrams:

$$\begin{tikzcd} \mathcal{M}\mathcal{M}(\underline{\mathcal{C}}) \arrow[r, "\mathcal{M}\delta"] \arrow[d, "\mu_{\underline{\mathcal{C}}}"'] & \mathcal{M}(\underline{\mathcal{C}}) \arrow[d, "\delta"] \\ \mathcal{M}(\underline{\mathcal{C}}) \arrow[r, "\delta"] & \underline{\mathcal{C}} \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathcal{C}} \arrow[rr, "\eta_{\underline{\mathcal{C}}}"] \arrow[dr, "id_{\underline{\mathcal{C}}}"'] & & \mathcal{M}(\underline{\mathcal{C}}) \arrow[dl, "\delta"] \\ & \underline{\mathcal{C}} & \end{tikzcd}$$

called the multiplication and the unit law for δ, which allows us to verify the axioms from Definition 1.4 as follows.

For the proof of (A1), let a and b be like above, let $c \in \underline{\mathcal{C}}(Z)$, $z \in Z$ and $u \in Y$. Suppose that a, b and c are all different from identity and that X, Y and Z are mutually disjoint (only to avoid the renaming technicalities). We will chase the multiplication diagram above two times, starting with two-level unrooted trees

$$\mathcal{T}_1 = \{[\{a(x,\dots), b(y,u,\dots); \sigma_1'\}]_\alpha, [\{c(z,\dots); id_Z\}]_\alpha; \sigma_1\}$$

and

$$\mathcal{T}_2 = \{[\{a(x,\dots); id_X\}]_\alpha, [\{b(y,u,\dots), c(z,\dots); \sigma_2'\}]_\alpha; \sigma_2\},$$

where $\sigma_1' = (x\ y)$, $\sigma_1 = (u\ z)$, $\sigma_2' = (u\ z)$ and $\sigma_2 = (x\ y)$. If we start with $\mathcal{T}_1$, then, by chasing the diagram to the right-down, the action of $\mathcal{M}\delta$ corresponds to the action of δ on $[\{a(x,\dots), b(y,u,\dots); \sigma_1'\}]_\alpha$ and $[\{c(z,\dots); id_Z\}]_\alpha$ separately. Followed by the action of δ again, we get the following sequence

$$\mathcal{T}_1 \xmapsto{\mathcal{M}\delta} [\{(a\,{}_x\circ_y\, b)(u,\dots), c(z,\dots); \sigma\}]_\alpha \xmapsto{\delta} (a\,{}_x\circ_y\, b)\ {}_u\circ_z\, c.$$

In the other direction, the action of the monad multiplication flattens $\mathcal{T}_1$, the resulting tree already being in normal form. Followed by the action of δ, we obtain the sequence:

$$\mathcal{T}_1 \xmapsto{\mu_{\underline{\mathcal{C}}}} [\{a(x,\dots), b(y,u,\dots), c(z,\dots); \underline{\sigma}\}]_\alpha \xmapsto{\delta} \delta([\{a(x,\dots), b(y,u,\dots), c(z,\dots); \underline{\sigma}\}]_\alpha).$$

Hence,

$$(a\,{}_x\circ_y\, b)\ {}_u\circ_z\, c = \delta([\{a(x,\dots), b(y,u,\dots), c(z,\dots); \underline{\sigma}\}]_\alpha).$$

The associativity follows since the diagram chasing with respect to $\mathcal{T}_2$ results in

$$a\,{}_x\circ_y\,(b\ {}_u\circ_z\, c) = \delta([\{a(x,\dots), b(y,u,\dots), c(z,\dots); \underline{\sigma}\}]_\alpha).$$

The axiom (CO) follows directly by the commutativity of ${}_x\bullet_y$.

The axiom (EQ) holds by the equivariance of ${}_x\bullet_y$ and the naturality of η and δ. For σ_1, σ_2 and σ as in (EQ), and denoting $Z = X'\backslash\{\sigma_1^{-1}(x)\} \cup Y'\backslash\{\sigma_2^{-1}(y)\}$, we have

$$\begin{aligned} a^{\sigma_1}\ {}_{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)}\ b^{\sigma_2} &= \delta_Z(\eta_{\mathcal{C}X'}(a^{\sigma_1})\ {}_{\sigma_1^{-1}(x)}\bullet_{\sigma_2^{-1}(y)}\ \eta_{\mathcal{C}Y'}(b^{\sigma_2})) \\ &= \delta_Z(\eta_X(a)^{\sigma_1}\ {}_{\sigma_1^{-1}(x)}\bullet_{\sigma_2^{-1}(y)}\ \eta_Y(b)^{\sigma_2}) \\ &= \delta_Z((\eta_X(a)\,{}_x\bullet_y\,\eta_Y(b))^\sigma) \\ &= \delta_Z(\eta_X(a)\,{}_x\bullet_y\,\eta_Y(b))^\sigma \\ &= (a\,{}_x\circ_y\, b)^\sigma. \end{aligned}$$

To prove (U1), we chase the multiplication law of δ, starting from the two-level tree

$$\mathcal{T} = \{[\{(y,z); id_{y,z}\}]_\alpha, [\{a(x,\dots); id_X\}]_\alpha; \sigma\},$$

where $\sigma = (x\ y)$. By going to the right-down, we get the sequence

$$\mathcal{T} \xmapsto{\mathcal{M}\delta} [\{id_{y,z}(y,z), a(x,\dots); \sigma\}]_\alpha \xmapsto{\delta} id_{y,z}\ {}_y\circ_x\, a,$$

and, in the other direction, denoting $X' = X\backslash\{x\} \cup \{z\}$, we get

$$\mathcal{T} \overset{\mu_{\underline{\mathcal{C}}}}{\longmapsto} [\{a^{\kappa}(z,\ldots); id_{X'}\}]_{\alpha} \overset{\delta}{\longmapsto} \delta([\{a^{\kappa}(z,\ldots); id_{X'}\}]_{\alpha}),$$

where κ renames x to z. The equality $id_{y,z}\, {}_y\circ_x a = a^{\kappa}$ follows, since, by the definition of the unit of the monad $\mathcal{M}$, and by the unit law of δ, for the result of the second sequence, we have

$$\delta([\{a^{\kappa}(z,\ldots); id_{X'}\}]_{\alpha}) = \delta(\eta_{\underline{\mathcal{C}}_{X'}}(a^{\kappa})) = a^{\kappa}.$$

For (UP), by the naturality of δ, for $\sigma : \{u,v\} \to \{x,y\}$ we have

$$\begin{aligned} id^{\sigma}_{x,y} &= \delta_{\{x,y\}}([\{(x,y); id_{\{x,y\}}\}]_{\alpha})^{\sigma} \\ &= \delta_{\{u,v\}}([\{(x,y); id_{\{x,y\}}\}]^{\sigma}_{\alpha}) \\ &= \delta_{\{u,v\}}([\{(u,v); id_{\{u,v\}}\}]_{\alpha}) \\ &= id_{u,v}. \end{aligned}$$

In the other direction, we define $\delta : \mathcal{M}(\underline{\mathcal{C}}) \to \underline{\mathcal{C}}$ as the map induced by the interpretation of the μ-syntax in the cyclic operad $\underline{\mathcal{C}}$, i.e. by the composition of $[[\_]] : \mu\texttt{Exp}_{\underline{\mathcal{C}}} \to \texttt{cTerm}_{\underline{\mathcal{C}}}$ and $[\_]_{\underline{\mathcal{C}}} : \texttt{cTerm}_{\underline{\mathcal{C}}} \to \underline{\mathcal{C}}$. Therefore, with Φ being defined as in the proof of Theorem 2.33, we set

$$\delta_X([\mathcal{T}]_{\alpha}) = [\,[[c]]\,]_{\underline{\mathcal{C}}}, \quad \text{where } c \text{ is any command of } \mu\texttt{Exp}_{\underline{\mathcal{C}}} \text{ such that } \underline{\Phi}(c) = \mathcal{T}.$$

Note that this definition is valid by Theorem 2.33. We verify that δ satisfies the equations of an $\mathcal{M}$-algebra on simple examples. The general case follows naturally. Let

$$\mathcal{T} = \{[\{a(x_1,\ldots,x_n), b(y_1,\ldots,y_m); \sigma_1\}]_{\alpha}, [\{d(z_1,\ldots,z_p); id_Z\}]_{\alpha}; \sigma\}$$

be a two-level unrooted tree such that $\sigma_1(x_i) = y_j$, and $\sigma(y_k) = z_l$, and suppose, say, that

$$\underline{\Phi}(\underline{a}\{t_1,\ldots,t_n\}) = [\{a(x_1,\ldots,x_n), b(y_1,\ldots,y_m); \sigma_1\}]_{\alpha}$$

and

$$\underline{\Phi}(\underline{d}\{s_1,\ldots,s_p\}) = [\{d(z_1,\ldots,z_p); id_Z\}]_{\alpha}.$$

By chasing the multiplication diagram to the right-down, the action of $\mathcal{M}\delta$ provides the interpretations of the commands that correspond to each of the corollas of $\mathcal{T}$. Thus, setting $[[\underline{a}\{t_1,\ldots,t_n\}]] = a(\varphi)$ and $[[\underline{d}\{s_1,\ldots,s_p\}]] = d(\tau)$, we get that

$$\mathcal{M}\delta([\mathcal{T}]_{\alpha}) = \{[a(\varphi)]_{\underline{\mathcal{C}}}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n,y_1,\ldots,y_{j-1},y_{j+1},\ldots,y_m), [d(\tau)]_{\underline{\mathcal{C}}}(z_1,\ldots,z_p); \sigma\}.$$

If now

$$\underline{\Phi}(\underline{[a(\varphi)]_{\underline{\mathcal{C}}}}\{k_1,\ldots,k_{n+m-2}\}) = \mathcal{M}\delta([\mathcal{T}]_{\alpha}),$$

then, by setting $[[\underline{[a(\varphi)]_{\underline{\mathcal{C}}}}\{k_1,\ldots,k_{n+m-2}\}]] = [a(\varphi)]_{\underline{\mathcal{C}}}(\psi)$, we get

$$\delta(\mathcal{M}\delta([\mathcal{T}]_{\alpha})) = [a(\varphi)(\psi)]_{\underline{\mathcal{C}}}.$$

By chasing the multiplicaiton diagram to the down-right, we first get

$$\mu_{\underline{\mathcal{C}}}([\mathcal{T}]_{\alpha}) = [\{a(x_1,\ldots,x_n), b(y_1,\ldots,y_m), d(z_1,\ldots,z_p); \underline{\sigma}\}]_{\alpha}$$

We shall construct a command c, such that $\underline{\Phi}(c) = \mu_{\underline{\mathcal{C}}}(\mathcal{T})$, in the way guided by the choices we made in chasing the diagram to the right-down. More precisely, in that direction, a was the corolla of $\{a(x_1,\ldots,x_n), b(y_1,\ldots,y_m); \sigma_1\}$ chosen in constructing the corresponding command, and d was the one for $\{d(z_1,\ldots,z_p); id_Z\}$, and then, in the next step, $[a(\varphi)]_{\underline{\mathcal{C}}}$ was the chosen corolla of $\mathcal{M}\delta([\mathcal{T}]_\alpha)$. Therefore, we set $c = \underline{a}\{\sigma\}$, where

$$\sigma(x_i) = \mu y_j.\underline{b}\{y_1, \ldots, y_{k-1}, \mu z_l.\underline{d}\{z_1, \ldots, z_p\}, y_{k+1}, \ldots, y_m\}.$$

Thus, setting $[[\underline{a}\{\sigma\}]] = a(\xi)$, we get

$$\delta(\mu_{\underline{\mathcal{C}}}(\mathcal{T})) = [a(\xi)]_{\underline{\mathcal{C}}}$$

as a result of chasing the diagram to the down-right. The equality $a(\varphi)(\psi) = a(\xi)$ follows by Lemma 2.1.(b).

As for the unit diagram, if $a \in \underline{\mathcal{C}}(X)$, where $X = \{x_1, \ldots, x_n\}$, then $\eta_{\underline{\mathcal{C}}_X}(a) = \{a(x_1, \ldots, x_n); id_X\}$, and, since $[\{a(x_1, \ldots, x_n); id_X\}]_\alpha = \underline{\Phi}(\underline{a}\{x_1, \ldots, x_n\})$, we have that

$$\delta_X(\eta_{\underline{\mathcal{C}}_X}(f)) = [\,[[\underline{a}\{x_1, \ldots, x_n\}]]\,]_{\underline{\mathcal{C}}} = a.$$

This completes the proof.

Chapter 3

Monoid-like definitions of cyclic operads

In this chapter, we transform the two biased definitions of cyclic operads, Definition 1.4 (with the alternative axiomatisation presented in Remark 1.8) and [Mar08, Proposition 42], into two algebraic definitions of the form

a cyclic operad is a monoid-like object in a certain monoidal-like category,

by following the microcosm principle behind the algebraic definitions of operads, established by Kelly [Kel05] and Fiore [Fio14]. Therefore, by the end of the chapter, we shall synthesise two tables of the same kind as Table 3 from the Introduction, which accompany and describe more explicitly the conceptual descriptions of the two algebraic definitons.

Residing in the category of Joyal's species of structures of the form $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$, the algebraic definitions that we deliver are crafted for non-skeletal characterisations of cyclic operads. Therefore, we first propose a non-skeletal version of [Mar08, Proposition 42]. We additionally give two proofs of the equivalence between the entries-only and exchangeable-output definitions (which, to the author's knowledge, has been taken for granted in the literature): one by comparing the usual biased definitions (Theorem 3.30), and the other by comparing two new algebraic definitions (Theorem 3.35). Together with the proof of the equivalence between biased and algebraic definitions of entries-only cyclic operads (Theorem 3.24), this makes a sequence of equivalences that also justifies the algebraic definition of exchangeable-output cyclic operads. An overview of the algebraic definitions that we introduce and the correspondences that we make is given in Table 4 below.

	Entries-only		Exchangeable-output
Biased	Definition 1.4	$\Longleftrightarrow$ Theorem 3.30	Definition 3.25
	Theorem 3.24 $\Updownarrow$		
Algebraic	Definition 3.19	Theorem 3.35 $\Longleftrightarrow$	Definition 3.31

Table 4: Algebraic definitions of cyclic operads

The plan of the chapter is as follows. Section 3.1 is an overview of the basic elements from the theory of species of structures. In Section 3.2, we recall the existing algebraic definitions of operads and indicate the microcosm principle behind them. Section 3.3 will be devoted to the introduction of the algebraic definitions of cyclic operads (Definition 3.19 and Definition 3.31) and of the biased non-skeletal version of [Mar08, Proposition 42] (Definition 3.25). Here we also prove the three theorems from Table 4.

Throughout this chapter, we shall work to a large extent with compositions of multiple canonical natural isomorphisms between functors. In order for such compositions not to look too cumbersome, we shall often omit their indices.

3.1 The category of species of structures

The content of this section is to a great extent a review and a gathering of material coming from [BLL08]. Certain isomorphisms, whose existence has been claimed in [BLL08], will be essential for subsequent sections and we shall construct them explicitely.

3.1.1 Definition of species of structures

The notion of species of structures that we fix as primary corresponds to functors underlying non-skeletal cyclic operads.

Definition 3.1. A species (of structures) is a functor $S : \mathbf{Bij}^{op} \to \mathbf{Set}$. □

In the remaining of the chapter, we shall refer to the functor category $\mathbf{Set}^{\mathbf{Bij}^{op}}$ as the *category of species* and we shall denote it with $\mathbf{Spec}$. For an arbitrary finite set X, an element $f \in S(X)$ will be referred to as an *S-structure*.

Notice that if S is a species and $\sigma : Y \to X$ is a bijection, then $S(\sigma) : S(X) \to S(Y)$ is necessarily a bijection (with the inverse $S(\sigma^{-1})$).

The following family of species will be essential for the treatement of operadic units in the subsequent sections.

EXAMPLE 3.2. The species E_n, where $n \geq 0$, called the *cardinality n species*, is defined by setting

$$E_n(X) = \begin{cases} \{X\} & \text{if } X \text{ has } n \text{ elements,} \\ \emptyset & \text{otherwise.} \end{cases}$$

□

An isomorphism between species is simply a natural isomorphism between functors. If there exists an isomorphism from S to T, we say that they S and T are isomorphic and we write $S \simeq T$.

3.1.2 Operations on species of structures

We now recall operations on species and their properties. Categorically speaking, every binary operation is a bifunctor of the form $\mathbf{Spec} \times \mathbf{Spec} \to \mathbf{Spec}$, every unary operation is a functor of the form $\mathbf{Spec} \to \mathbf{Spec}$ and every property of an operation holds up to isomorphism of species.

We start with the analogues of the arithmetic operations of addition and multiplication.

Definition 3.3. Let S and T be species, X an arbitrary finite set and $\sigma : Y \to X$ a bijection. The *sum-species of S and T* is the species $S + T$, defined by

$$(S + T)(X) = S(X) + T(X)$$

and

$$(S + T)(\sigma)(f) = \begin{cases} S(\sigma)(f) & \text{if } f \in S(X) \\ T(\sigma)(f) & \text{if } f \in T(X)\,. \end{cases}$$

The *product-species of S and T* is the species $S \cdot T$ defined by

$$(S \cdot T)(X) = \sum_{(X_1, X_2)} S(X_1) \times T(X_2),$$

where the sum is taken over all binary decompositions (X_1, X_2) of X. The action of $S \cdot T$ on σ is given as

$$(S \cdot T)(\sigma)(f, g) = (S(\sigma_1)(f), T(\sigma_2)(g)),$$

where $\sigma_i = \sigma|^{X_i}$, $i = 1, 2$. □

The isomorphisms from the following lemma are constructed straightforwardly.

Lemma 3.4. *The addition and multiplication of species have the following properties.*

1. *The operation of addition is associative and commutative.*
2. *The product of species is associative and commutative. The cardinality* 0 *species* E_0 *is neutral element for the product of species. Therefore, for all species* S, $S \cdot E_0 \simeq E_0 \cdot S \simeq S$.

Notation 3.5. *We extend the notation* f_1+f_2 *and* $[f_1, f_2]$ *(see the paragraph "Notations and conventions" in Introduction) from functions to natural transformations. For natural transformations* $\psi_i : S_i \rightarrow T_i$, $i = 1, 2$, $\psi_1+\psi_2 : S_1+S_2 \rightarrow T_1+T_2$ *will denote the natural transformation determined by* $(\psi_1+\psi_2)_X = {\psi_1}_X + {\psi_2}_X$. *For natural transformations* $\kappa_i : S_i \rightarrow U$, $i = 1, 2$, $[\kappa_1, \kappa_2] : S_1 + S_2 \rightarrow U$ *will denote the natural transformation defined as* $[\kappa_1, \kappa_2]_X = [{\kappa_1}_X, {\kappa_2}_X]$. *With* i_l *and* i_r *we shall denote the insertion natural transformations* $i_l : S \rightarrow S + T$ *and* $i_r : T \rightarrow S + T$*, respectively.*

We recall next the operation corresponding to the operation of substitution.

Definition 3.6. Let S and T be species and X an arbitrary finite set. The *substitution product of* S *and* T is the species $S \circ T$ defined by

$$(S \circ T)(X) = \sum_{P} \sum_{\phi : X \rightarrow P} \left(S(P) \times \prod_{p \in P} T(\phi^{-1}(p)) \right) \Big/ _{\simeq},$$

where P ranges over finite sets, $\phi : X \rightarrow P$ ranges over (arbitrary) functions (from X to P), and $\simeq$ is the smallest equivalence relation generated by

$$(P, \phi, f, (g_p)_{p \in P}) \simeq (P', \tau \circ \phi, S(\tau)(f), (g_{\tau^{-1}(p')})_{p' \in P'}),$$

where $\tau : P \rightarrow P'$ is an arbitrary bijection. Note that ϕ is not surjective in general, i.e. that a fiber $\phi^{-1}(p)$ may be the empty set for some $p \in P$. For a bijection $\sigma : Y \rightarrow X$ and $h = (P, \phi, f, (g_p)_{p \in P}) \in (S \circ T)(X)$, the action of $S \circ T$ on σ is defined by

$$(S \circ T)(\sigma)(h) = (P, \phi \circ \sigma, f, (\overline{g}_p)_{p \in P}),$$

where $\overline{g}_p = T(\sigma|^{\phi^{-1}(p)})(g_p)$. $\square$

Basic properties of the substitution product are given in the following lemma.

Lemma 3.7. *The substitution product of species is associative and has the cardinality 1 species* E_1 *as neutral element.*

Next comes the analogue of the operation of derivation.

Definition 3.8. The *derivative of* S is the species ∂S, defined by

$$(\partial S)(X) = S(X \cup \{*_X\}),$$

where $*_X \notin X$. The action of ∂S on σ is defined by

$$(\partial S)(\sigma)(f) = S(\sigma^+)(f),$$

where $\sigma^+ : Y \cup \{*_Y\} \rightarrow X \cup \{*_X\}$ is such that $\sigma^+(y) = \sigma(y)$ for $y \in Y$ and $\sigma^+(*_Y) = *_X$. We shall refer to σ^+ as the *∂-extension of* σ. $\square$

We now introduce a natural isomorphism that will be used for the algebraic version of the associativity axiom for entries-only cyclic operads. Let $f \in \partial\partial S(X)$ and let

$$\varepsilon_X : X \cup \{*_X, *_{X \cup \{*_X\}}\} \rightarrow X \cup \{*_X, *_{X \cup \{*_X\}}\}$$

be the bijection which acts as the identity on X and such that $\varepsilon_X(*_X) = *_{X\cup\{*_X\}}$ (and $\varepsilon_X(*_{X\cup\{*_X\}}) = *_X$). We define a natural transformation $\mathtt{ex}_S : \partial(\partial S) \to \partial(\partial S)$ by setting

$$\mathtt{ex}_{S\,X}(f) = S(\varepsilon_X)(f).$$

We shall refer to $\mathtt{ex}_S$ as the *exchange isomorphism*, since its components exchange the two distinguished elements arising from the two-fold application of the operation of derivation.

The following lemma exibits isomorphisms between species that correspond to the rules of *the derivative of a sum* and *the derivative of a product* of the classical differential calculus.

Lemma 3.9. *For arbitrary species S and T, the following properties hold:*

1. $\partial(S+T) \simeq \partial S + \partial T$, *and*
2. $\partial(S\cdot T) \simeq (\partial S)\cdot T + S\cdot(\partial T)$.

Proof. The isomorphism $\varsigma : \partial(S+T) \to \partial S + \partial T$, establishing the first property, is the identity natural transformation.

For the second property, we define an isomorphism $\varphi : \partial(S\cdot T) \to (\partial S)\cdot T + S\cdot(\partial T)$. For a finite set X, we have

$$\begin{array}{rcl}\partial(S\cdot T)(X) & = & \sum_{(X_1,X_2)}\{(f,g)\,|\, f\in S(X_1) \text{ and } g\in T(X_2)\},\\ (\partial S\cdot T)(X) & = & \sum_{(X_1',X_2')}\{(f,g)\,|\, f\in (\partial S)(X_1') \text{ and } g\in T(X_2')\}, \text{ and}\\ (S\cdot \partial T)(X) & = & \sum_{(X_1',X_2')}\{(f,g)\,|\, f\in S(X_1') \text{ and } g\in (\partial T)(X_2')\},\end{array}$$

where (X_1,X_2) is an arbitrary binary decomposition of the set $X\cup\{*_X\}$, and (X_1',X_2') is an arbitrary binary decomposition of the set X.

If $(f,g)\in\partial(S\cdot T)(X)$, where $f\in S(X_1)$ and $g\in T(X_2)$, and if $*_X\in X_1$, then $(X_1',X_2') = (X_1\backslash\{*_X\},X_2)$ is a binary decomposition of the set X and we set

$$\varphi_X(f,g) = (S(\sigma)(f),g),$$

where $\sigma : X_1\backslash\{*_X\}\cup\{*_{X_1'}\} \to X_1$ renames $*_X$ to $*_{X_1'}$. We do analogously if $*_X\in X_2$.

To define the inverse of φ_X, suppose that $(f,g)\in(\partial S\cdot T)(X)$, where $f\in(\partial S)(X_1')$ and $g\in T(X_2')$. The pair $(X_1'\cup\{*_{X_1'}\},X_2')$ is then a binary decomposition of the set $X\cup\{*_{X_1'}\}$. Let $\tau : X_1'\cup\{*_X\}\to X_1'\cup\{*_{X_1'}\}$ be the renaming of $*_{X_1'}$ to $*_X$. The pair $(X_1,X_2) = (X_1'\cup\{*_X\},X_2')$ is now a binary decomposition of the set $X\cup\{*_X\}$ and we set

$$\varphi_X^{-1}(f,g) = (S(\tau)(f),g)\in\partial(S\cdot T)(X).$$

We proceed analogously for $(f,g)\in(S\cdot\partial T)(X)$. ■

We shall also need the family of isomorphisms from the following lemma.

Lemma 3.10. *For all $n\geq 1$, $\partial E_n \simeq E_{n-1}$.*

Proof. For a finite set X we have

$$\partial E_n(X) = \begin{cases}\{X\cup\{*_X\}\} & \text{if } |X| = n-1,\\ \emptyset & \text{otherwise,}\end{cases}$$

and

$$E_{n-1}(X) = \begin{cases}\{X\} & \text{if } |X| = n-1,\\ \emptyset & \text{otherwise.}\end{cases}$$

The isomorphism $\epsilon_n : \partial E_n \to E_{n-1}$ is defined by $\epsilon_{nX}(X \cup \{*_X\}) = X$, for $|X| = n-1$. Otherwise, ϵ_{nX} is the empty function. ■

Finally, we shall also use the following *pointing* operation on species.

Definition 3.11. Let S be a species. The species $S^\bullet$, spelled *S dot*, is defined as follows

$$S^\bullet(X) = S(X) \times X.$$

For a pair $(f, x) \in S(X) \times X$, the action of $S^\bullet$ on a bijection $\sigma : Y \to X$ is given by

$$S^\bullet(\sigma)((f, x)) = (S(\sigma)(f), \sigma^{-1}(x)).$$

□

Remark 3.12. *Observe that the distinguished element of an $S^\bullet$-structure belongs to the underlying set X, as opposed to the distinguished element of a ∂S-struture, which is always outside of X.*

To summarise, we list in Table 5 the isomorphisms between species that we shall use in the remaining of the chapter.

Name	Reference	Description
associativity of $\cdot$	$\alpha_{S,T,U} : (S \cdot T) \cdot U \to S \cdot (T \cdot U)$	$((f, g), h) \mapsto (f, (g, h))$
commutativity of $\cdot$	$\gamma_{S,T} : S \cdot T \to T \cdot S$	$(f, g) \mapsto (g, f)$
left unitor for $\cdot$	$\lambda_S : E_0 \cdot S \to S$	$(\{\emptyset\}, f) \mapsto f$
right unitor for $\cdot$	$\rho_S : S \cdot E_0 \to S$	$(f, \{\emptyset\}) \mapsto f$
exchange	$\mathtt{ex}_S : \partial(\partial S) \to \partial(\partial S)$	$f \mapsto S(\varepsilon)(f)$
derivative of a sum	$\varsigma_{S,T} : \partial(S + T) \to \partial S + \partial T$	Lemma 3.9 (1)
Leibniz rule	$\varphi_{S,T} : \partial(S \cdot T) \to (\partial S) \cdot T + S \cdot (\partial T)$	Lemma 3.9 (2)
ϵ_n-isomorphism	$\epsilon_n : \partial E_n \to E_{n-1}$	Lemma 3.10

Table 5: Canonical isomorphisms between species

3.2 Algebraic definitions of operads

This part is a reminder on algebraic definitions of operads. Our emphasis is on the use of the microcosm principle of Baez and Dolan, which we illustrate by reviewing Fiore's definition in §3.2.2 below.

3.2.1 Kelly-May definition

Kelly's monoidal definition [Kel05, Section 4] is the algebraic version of the original definition of an operad [May72] of May. In the non-skeletal setting, the operadic composition of May's definition is given by morphisms

$$\gamma_{X,Y_1,\ldots,Y_n} : S(X) \times S(Y_1) \times \cdots \times S(Y_n) \to S(Y_1 \cup \cdots \cup Y_n)\,, \tag{3.2.1}$$

defined for a finite set X and pairwise disjoint finite sets $Y_1, \ldots, Y_n$, where $n = |X|$, and the unit $id_x \in S(\{x\})$, defined for all singletons $\{x\}$, which are subject to associativity, equivariance and unit axioms.

In order to arrive to Kelly's definition, one first observes that Lemma 3.7 can easily be reinforced to a stronger claim:

$(\mathbf{Spec}, \circ, E_1)$ *is a monoidal category.*

A monoid in this category is a triple (S, μ, η), where S is a species and the natural transformations $\mu : S \circ S \to S$ and $\eta : E_1 \to S$, called the *multiplication* and the *unit* of the monoid, respectively, satisfiy the coherence conditions given by the commutation of the following two diagrams

$$\begin{tikzcd} (S\circ S)\circ S \arrow[r, "\alpha^\circ"] \arrow[d, "\mu\circ id"'] & S\circ(S\circ S) \arrow[r, "id\circ\mu"] & S\circ S \arrow[d, "\mu"] \\ S\circ S \arrow[rr, "\mu"'] & & S \end{tikzcd} \qquad \begin{tikzcd} E_1\circ S \arrow[r, "\eta\circ id"] \arrow[dr, "\lambda^\circ"'] & S\circ S \arrow[d, "\mu"] & S\circ E_1 \arrow[l, "id\circ\eta"'] \arrow[dl, "\rho^\circ"] \\ & S & \end{tikzcd}$$

in which α°, λ° and ρ° denote the associator, left and right unitor of $(\mathbf{Spec}, \circ, E_1)$, respectively.

Note that an element $(f, g_1, \ldots, g_n) \in S(X) \times S(Y_1) \times \cdots \times S(Y_n)$ determines the element

$$[(X, \phi : Y_1 \cup \cdots \cup Y_n \to X, f, (g_i)_{1\leq i\leq n})]_{\simeq} \in (S \circ S)(Y_1 \cup \cdots \cup Y_n).$$

By defining the simultaneous composition operation (3.2.1) as

$$\gamma_{X,Y_1,\ldots Y_n}(f, g_1, \ldots, g_n) = \mu([(X, \phi, f, g_1, \ldots, g_n)]_{\simeq}),$$

and id_x as $\eta_{\{x\}}(\{x\})$, the operadic axioms are easily verified by the naturality of μ and laws of the monoid. This gives a crisp alternative to the somewhat cumbersome biased definition:

An operad is a monoid in the monoidal category $(\mathbf{Spec}, \circ, E_1)$.

The steps to derive the monoidal definition from above and, more generally, a monoid-*like* definition of an arbitrary operad-like structure, starting from its biased characterisation, can be summarised as follows. One first has to exhibit a product $\diamond$ on $\mathbf{Spec}$ that captures the type of operadic composition that is to be formalised (in the same way as the representative $(X, \phi, f, (g_i)_{1\leq i\leq n})$ of the appropriate $(S \circ S)$-structure corresponds to the configuration $(f, g_1, \ldots, g_n)$ of operadic operations). One then has to examine the properties of this product, primarily by comparing species $(S \diamond T) \diamond U$ and $S \diamond (T \diamond U)$, in order to exhibit an isomorphism whose commutation with the multiplication $\mu : S \diamond S \to S$ expresses axioms of the operad-like structure in question. Analogously, an appropriate isomorphism of species is needed for each of the remaining axioms of such a structure (for example, the isomorphims λ°_S and ρ°_S account for the unit axioms of an operad), except for the equivariance axiom, which holds by the naturality of μ. The operad-like structure is then introduced as an object S of $\mathbf{Spec}$, together with the multiplication μ (and possibly other natural transformations, like the unit η in the previous definition) that commutes in the appropriate way with established isomorphisms.

3.2.2 Fiore-Markl definition

We now recover Fiore's algebraic definition (see [Fio12] and [Fio14]), established for operads with partial composition of Definition 1.4, by following the steps described in the last paragraph of §3.2.1.

Regarding Definition 1.4, the data out of which the composition $f \circ_x g$ is obtained consists of the ordered pair (f, g), together with a chosen input x of f. This indicates that the product of species that is supposed to capture partial composition operation should involve the product $\cdot : \mathbf{Spec} \times \mathbf{Spec} \to \mathbf{Spec}$ introduced by Definition 3.3, whereby the structures arising from the left component of the product should have a distinguished element among the elements of the underlying set. Hence, a priori, there are two possible candidates for the new product:

$$S^\bullet \cdot S \quad \text{and} \quad (\partial S) \cdot S.$$

However, the first one does not work: for $(f, g) \in (S^\bullet \cdot S)(X)$, the multiplication $(S^\bullet \cdot S)(X) \to S(X)$ produces an element of $S(X)$, whereas the composition $f \circ_x g$ should belong to $S(X\backslash\{x\})$. On the other hand, the elements of the set $(\partial S \cdot S)(X)$ are pairs (f, g) such that $f \in S(X_1 \cup \{*_{X_1}\})$ and $g \in S(X_2)$, where (X_1, X_2) is a decomposition of the set X. From the operadic perspective, the composition of $f \circ_{*_{X_1}} g$ belongs to $S(X)$, which agrees with the form of the multiplication $\nu_X : (\partial S \cdot S)(X) \to S(X)$. Therefore, as a tentative product of species we take the *pre-Lie product* $S \star T$, defined as

$$S \star T = \partial S \cdot T.$$

The next step is to compare the species $(S \star T) \star U$ and $S \star (T \star U)$. Chasing the associativity fails in this case. However, there is a canonical natural *pre-Lie isomorphism*

$$\beta_{S,T,U} : (S \star T) \star U + S \star (U \star T) \to S \star (T \star U) + (S \star U) \star T$$

determined by the isomorphisms

$$\begin{aligned} &\beta_1 : (\partial\partial S \cdot T) \cdot U \to (\partial\partial S \cdot U) \cdot T && \beta_1 = \alpha^{-1} \circ (\mathtt{ex} \cdot \gamma) \circ \alpha, \\ &\beta_2 : (\partial S \cdot \partial T) \cdot U \to \partial S \cdot (\partial T \cdot U) && \beta_2 = \alpha, \\ &\beta_3 : \partial S \cdot (\partial U \cdot T) \to (\partial S \cdot \partial U) \cdot T && \beta_3 = \alpha^{-1}, \end{aligned}$$

where α, γ and ex stand for appropriate instances of isomorphisms given in Table 5. The pre-Lie isomorphism is the "smallest" isomorphism that captures both associativity axioms for operads (β_1 accounts for [A1] and β_2 for [A2]).

For the algebraic account on operadic units, we shall use the isomorphisms exhibited in the following lemma.

Lemma 3.13. *For an arbitrary species S, $E_1 \star S \simeq S$ and $S \star E_1 \simeq S^\bullet$.*

Proof. The isomorphism $\lambda^\star_S : E_1 \star S \to S$ arises as

$$E_1 \star S = \partial E_1 \cdot S \simeq E_0 \cdot S \simeq S,$$

i.e. as $\lambda^\star_S = \lambda^{\cdot}_S \circ (\epsilon_1 \cdot id_S)$. We postpone the definition of the isomorphism $\rho^\star_S : S \star E_1 \to S^\bullet$ for Remark 3.18. ■

By following the microcosm principle heuristics, from these data Fiore induced the following definition, in which we write β for $\beta_{S,S,S}$.

Definition 3.14. An operad is a triple (S, ν, η_1) of a species S, a natural transformation $\nu : S \star S \to S$, called the *multiplication*, and a natural transformation $\eta_1 : E_1 \to S$, called the *unit*, such that

[OA1] $\nu_2 \circ \beta = \nu_1$, where ν_1 and ν_2 are induced by ν as follows:

- $\nu_1 : (S \star S) \star S + S \star (S \star S) \to S$ is determined by

$$\begin{aligned} &\nu_{11} : (\partial\partial S \cdot S) \cdot S \xrightarrow{i_l \cdot id} (\partial\partial S \cdot S + \partial S \cdot \partial S) \cdot S \xrightarrow{\varphi^{-1} \cdot id} \partial(\partial S \cdot S) \cdot S \xrightarrow{\partial\nu \cdot id} \partial S \cdot S \xrightarrow{\nu} S, \\ &\nu_{12} : (\partial S \cdot \partial S) \cdot S \xrightarrow{i_r \cdot id} (\partial\partial S \cdot S + \partial S \cdot \partial S) \cdot S \xrightarrow{\varphi^{-1} \cdot id} \partial(\partial S \cdot S) \cdot S \xrightarrow{\partial\nu \cdot id} \partial S \cdot S \xrightarrow{\nu} S, \\ &\nu_{13} : \partial S \cdot (\partial S \cdot S) \xrightarrow{id \cdot \nu} \partial S \cdot S \xrightarrow{\nu} S, \end{aligned}$$

- $\nu_2 : S \star (S \star S) + (S \star S) \star S \to S$ is determined by $\nu_{21} = \nu_{11}$, $\nu_{22} = \nu_{13}$ and $\nu_{23} = \nu_{12}$, and

[OA2] η_1 satisfies coherence conditions given by the commutation of the following diagram

$$\begin{tikzcd} E_1 \star S \arrow[r, "\eta_1 \star id"] \arrow[dr, "\lambda^\star"'] & S \star S \arrow[d, "\nu"] & S \star E_1 \arrow[l, "id \star \eta_1"'] \arrow[d, "\rho^\star"] \\ & S & S^\bullet \arrow[l, "\pi_1"'] \end{tikzcd}$$

□

Indeed, it can be shown that `[OA1]` acounts for `[A1]` and `[A2]`[1], the naturality of ν ensures `[EQ]`, `[OA2]` proves `[U1]` and `[U2]`, are the naturality of η ensures `[UP]`.

3.3 Algebraic definitions of cyclic operads

This section contains the algebraic treatment of cyclic operads. The first part deals with the algebraic counterpart of Definition 1.4, i.e., strictly speaking, of its equivalent specification, determined by axioms

`(A2)`, `(EQ)`, `(U1)` and `(UP)`

(see Remark 1.8). In the second part, we first set up the non-skeletal exchangeable-output definition of cyclic operads, and then deliver its algebraic counterpart.

3.3.1 Entries-only

Applying the microcosm principle starting from Definition 1.4 begins with the observation that the data out of which the composition $f\,{}_x{\circ}_y\,g$ is obtained consists of the pair (f, g) and chosen entries x and y of f and g, respectively. The discussion we had for operads in §3.2.2 makes it easy to guess which combination of the product and derivative of species is the right one in this case.

Definition 3.15. Let S and T be species. The *triangle product* (or, shorter, the *▲-product*) *of* S *and* T is the species $S\blacktriangle T$ defined by

$$S\blacktriangle T = \partial S \cdot \partial T.$$

□

By "unfolding" the definition of the triangle product, we see that, for a finite set X,

$$(S\blacktriangle T)(X) = \sum_{(X_1,X_2)} \{(f,g) \mid f \in S(X_1 \cup \{*_{X_1}\}), g \in T(X_2 \cup \{*_{X_2}\})\},$$

and, for $(f, g) \in (S\blacktriangle T)(X)$ and a bijection $\sigma : Y \to X$,

$$(S\blacktriangle T)(\sigma)(f,g) = (S(\sigma_1^+)(f), T(\sigma_2^+)(g)),$$

where $\sigma_1 = \sigma|^{X_1}$, $\sigma_2 = \sigma|^{X_2}$, and σ_i^+ are the ∂-extensions of σ_i, $i = 1, 2$.

Remark 3.16. *The isomorphism* $\gamma_{\partial S,\partial T} : \partial S \cdot \partial T \to \partial T \cdot \partial S$ *(see Table 5) witnesses the commutativity of ▲-product.*

The next step is to exhibit an isomorphism that equates various ways to derive a ▲-product of three species. Intuitively, in the language of species, the associativity axiom `(A2)` can be stated as the existence of an isomorphism of the form $(\partial\partial S \cdot \partial T) \cdot \partial U \to (\partial\partial S \cdot \partial U) \cdot \partial T$. It turns out that the "smallest" isomorphism that compares $(S\blacktriangle T)\blacktriangle U$ and $S\blacktriangle(T\blacktriangle U)$ and includes the above isomorphism is

$$\theta_{S,T,U} : (S\blacktriangle T)\blacktriangle U + T\blacktriangle(S\blacktriangle U) + (T\blacktriangle U)\blacktriangle S \to S\blacktriangle(T\blacktriangle U) + (S\blacktriangle U)\blacktriangle T + U\blacktriangle(S\blacktriangle T)$$

[1] Actually, the equalities $\nu_{21} \circ \beta = \nu_{11}$ and $\nu_{22} \circ \beta = \nu_{12}$ are enough to prove associativity.

whose explict description is the following. By "unfolding" the definition of $\blacktriangle$, we see that $\theta_{S,T,U}$ connects a sum of 6 species on the left with a sum of 6 species on the right. Here is the list of the 6 constituents of $\theta_{S,T,U}$, together with their explicit definitions:

$$\begin{array}{ll}
\theta_1 : (\partial\partial S \cdot \partial T) \cdot \partial U \to (\partial\partial S \cdot \partial U) \cdot \partial T & \theta_1 = \alpha^{-1} \circ (\mathtt{ex} \cdot \gamma) \circ \alpha, \\
\theta_2 : (\partial S \cdot \partial\partial T) \cdot \partial U \to \partial S \cdot (\partial\partial T \cdot \partial U) & \theta_2 = \gamma \circ \theta_1 \circ (\gamma \cdot id), \\
\theta_3 : \partial T \cdot (\partial\partial S \cdot \partial U) \to \partial U \cdot (\partial\partial S \cdot \partial T) & \theta_3 = \gamma \circ \theta_1 \circ \gamma, \\
\theta_4 : \partial T \cdot (\partial S \cdot \partial\partial U) \to \partial S \cdot (\partial T \cdot \partial\partial U) & \theta_4 = (id \cdot \gamma) \circ \gamma \circ \theta_1 \circ (\gamma \cdot id) \circ \gamma, \\
\theta_5 : (\partial\partial T \cdot \partial U) \cdot \partial S \to \partial U \cdot (\partial S \cdot \partial\partial T) & \theta_5 = (id \cdot \gamma) \circ \gamma \circ \theta_1, \text{ and} \\
\theta_6 : (\partial T \cdot \partial\partial U) \cdot \partial S \to (\partial S \cdot \partial\partial U) \cdot \partial T & \theta_6 = \gamma \circ (id \cdot \gamma) \circ \gamma \circ \theta_1 \circ (\gamma \cdot id).
\end{array}$$

Notice that, having fixed θ_1, the pairing given by θ_3 is also predetermined, but there are other ways to pair the remaining 4 summands from the left with the 4 summands from the right. We made this particular choice in order for all θ_i to represent "parallel associativity modulo commutativity" (see Lemma 1.6), but a different pairing could have been chosen as well.

What remains is to exhibit the structure on species that will account for operadic units. The following lemma is essential.

Lemma 3.17. *For an arbitrary species S, $E_2\blacktriangle S \simeq S^\bullet$ and $S\blacktriangle E_2 \simeq S^\bullet$.*

Proof. By the definition of the $\blacktriangle$-product and of the species E_2, we have

$$(E_2\blacktriangle S)(X) = \sum_{x\in X}\{(\{x, *_{\{x\}}\}, f)\,|\, , f \in S(X\backslash\{x\} \cup \{*_{X\backslash\{x\}}\})\}. \tag{3.3.1}$$

We define $\lambda^\blacktriangle_S : E_2\blacktriangle S \to S^\bullet$ as

$$\lambda^\blacktriangle_{S\,X} : (\{x, *_{\{x\}}\}, f) \mapsto (S(\sigma)(f), x),$$

where $\sigma : X \to X\backslash\{x\} \cup \{*_{X\backslash\{x\}}\}$ renames $*_{X\backslash\{x\}}$ to x. For $X = \emptyset$, $\lambda^\blacktriangle_{S\,X}$ is the empty function. The isomorphism $\kappa^\blacktriangle_S : S\blacktriangle E_2 \to S^\bullet$ is defined by $\kappa^\blacktriangle_S = \lambda^\blacktriangle_S \circ \gamma_{\partial S,\partial E_2}$. ∎

Remark 3.18. *Since $\partial E_2 \simeq E_1$, we also have that $E_1 \cdot \partial S \simeq S^\bullet$. The isomorphism $\zeta_S : S^\bullet \to E_1 \cdot \partial S$ is defined as $(\epsilon_2^{-1} \cdot id_{\partial S})^{-1} \circ (\lambda^\blacktriangle_S)^{-1}$, i.e. for $f \in S(X)$ and $x \in X$, we have*

$$\zeta_{S\,X}(f, x) = (\{x\}, S(\sigma)(f)),$$

where $\sigma : X\backslash\{x\} \cup \{*_{X\backslash\{x\}}\} \to X$ renames x to $*_{X\backslash\{x\}}$. Going back to the proof of Lemma 3.13, for the definition of $\rho^\star_S$ we take preciesly $\zeta_S^{-1} \circ \gamma_{\partial S,E_1}$.

These data are assembled by the microcosm principle as follows. In the definition below (and further on in this chapter), we shall denote $\theta_{S,S,S}$ simply with θ.

Definition 3.19. An entries-only cyclic operad is a triple (S, ρ, η_2) of a species S, a natural transformation $\rho : S\blacktriangle S \to S$, called the *multiplication*, and a natural transformation $\eta_2 : E_2 \to S$, called the *unit*, such that

(CA1) $\rho_2 \circ \theta = \rho_1$, where ρ_1 and ρ_2 are induced from ρ as follows:

- $\rho_1 : (S\blacktriangle S)\blacktriangle S + S\blacktriangle(S\blacktriangle S) + (S\blacktriangle S)\blacktriangle S \to S$ is determined by

$$\begin{array}{l}
\rho_{11} : (\partial\partial S \cdot \partial S) \cdot \partial S \xrightarrow{i_l \cdot id} (\partial\partial S \cdot \partial S + \partial S \cdot \partial\partial S) \cdot \partial S \xrightarrow{\varphi^{-1} \cdot id} \partial(\partial S \cdot \partial S) \cdot \partial S \xrightarrow{\partial\rho \cdot id} \partial S \cdot \partial S \xrightarrow{\rho} S \\
\rho_{12} : (\partial S \cdot \partial\partial S) \cdot \partial S \xrightarrow{i_r \cdot id} (\partial\partial S \cdot \partial S + \partial S \cdot \partial\partial S) \cdot \partial S \xrightarrow{\varphi^{-1} \cdot id} \partial(\partial S \cdot \partial S) \cdot \partial S \xrightarrow{\partial\rho \cdot id} \partial S \cdot \partial S \xrightarrow{\rho} S \\
\rho_{13} : \partial S \cdot (\partial\partial S \cdot \partial S) \xrightarrow{id \cdot i_l} \partial S \cdot (\partial\partial S \cdot \partial S + \partial S \cdot \partial\partial S) \xrightarrow{id \cdot \varphi^{-1}} \partial S \cdot \partial(\partial S \cdot \partial S) \xrightarrow{id \cdot \partial\rho} \partial S \cdot \partial S \xrightarrow{\rho} S \\
\rho_{14} : \partial S \cdot (\partial S \cdot \partial\partial S) \xrightarrow{id \cdot i_r} \partial S \cdot (\partial\partial S \cdot \partial S + \partial S \cdot \partial\partial S) \xrightarrow{id \cdot \varphi^{-1}} \partial S \cdot \partial(\partial S \cdot \partial S) \xrightarrow{id \cdot \partial\rho} \partial S \cdot \partial S \xrightarrow{\rho} S \\
\rho_{15} = \rho_{11} \quad \text{and} \quad \rho_{16} = \rho_{12}
\end{array}$$

- $\rho_2 : S\blacktriangle(S\blacktriangle S) + (S\blacktriangle S)\blacktriangle S + S\blacktriangle(S\blacktriangle S) \to S$ is determined by $\rho_{21} = \rho_{11}$, $\rho_{22} = \rho_{23} = \rho_{13}$, $\rho_{24} = \rho_{25} = \rho_{14}$ and $\rho_{26} = \rho_{12}$, and

(CA2) η_2 satisfies the coherence condition given by the commutation of the diagram

$$\begin{tikzcd}
E_2\blacktriangle S \arrow[r, "\eta_2\blacktriangle id"] \arrow[d, "\lambda^{\blacktriangle}_S"'] & S\blacktriangle S \arrow[d, "\rho"] \\
S^{\bullet} \arrow[r, "\pi_{1S}"'] & S
\end{tikzcd}$$

where $\lambda^{\blacktriangle}_S$ is the isomorphisms from Lemma 3.17, and π_{1SX} is the first projection. □

In the following lemma we prove the equality that represents the algebraic analogue of the commutativity law (CO), which would follow anyhow after we prove the equivalence between Definition 1.4 and Definition 3.19. Nevertheless, we do this directly since it shortens significantly the proof of that equivalence.

Lemma 3.20. *For an arbitrary entries-only cyclic operad (S, ρ, η_2), the equality $\rho \circ \gamma = \rho$ holds.*

Proof. In Diagram 1 below, D_l and D_r commute by (CA2), D_t and D_m commute by the naturality of θ_1, and D_b commutes as it represents the equality $\rho_{21} \circ \theta_1 = \rho_{11}$.

For a finite set X, let $(f, g) \in (\partial S \cdot \partial S)(X)$, where $f \in \partial S(X_1)$, $y \in \partial S(X_2)$ and (X_1, X_2) is an arbitrary decomposition of X. Starting with (f, g), we chase Diagram 1 from the top left node $\partial S \cdot \partial S$ to the bottom left node $\partial S \cdot \partial S$, by going through the left "border" of the diagram, i.e. by applying the composition

$$(\partial\pi_1 \circ \partial\lambda^{\blacktriangle} \circ \varphi^{-1} \circ i_l \circ (\partial\epsilon_2^{-1} \cdot id) \circ \lambda^{\star -1}) \cdot id. \tag{3.3.2}$$

$$\begin{tikzcd}
 & \partial S \cdot \partial S \arrow[rr, "\gamma"] \arrow[d, "((\partial\epsilon_2^{-1} \cdot id) \circ \lambda^{\star -1}) \cdot id"'] & \arrow[d, phantom, "D_t"] & \partial S \cdot \partial S \arrow[d, "((\partial\epsilon_2^{-1} \cdot id) \circ \lambda^{\star -1}) \cdot id"] & \\
\partial(\partial E_2 \cdot \partial S) \cdot \partial S \arrow[dd, "\partial\lambda^{\blacktriangle} \cdot id"'] & (\partial\partial E_2 \cdot \partial S) \cdot \partial S \arrow[l, "(\varphi^{-1} \circ i_l) \cdot id"'] \arrow[rr, "\theta_1"] \arrow[d, "(\partial\partial\eta_2 \cdot id) \cdot id"'] & \arrow[d, phantom, "D_m"] & (\partial\partial E_2 \cdot \partial S) \cdot \partial S \arrow[r, "(\varphi^{-1} \circ i_l) \cdot id"] \arrow[d, "(\partial\partial\eta_2 \cdot id) \cdot id"] & \partial(\partial E_2 \cdot \partial S) \cdot \partial S \arrow[dd, "\partial\lambda^{\blacktriangle} \cdot id"] \\
\arrow[r, phantom, "D_l"] & (\partial\partial S \cdot \partial S) \cdot \partial S \arrow[rr, "\theta_1"] \arrow[d, "(\partial\rho \circ \varphi^{-1} \circ i_l) \cdot id"'] & \arrow[d, phantom, "D_b"] & (\partial\partial S \cdot \partial S) \cdot \partial S \arrow[d, "(\partial\rho \circ \varphi^{-1} \circ i_l) \cdot id"] \arrow[r, phantom, "D_r"] & \\
\partial(S^{\bullet}) \cdot \partial S \arrow[r, "\partial\pi_1 \cdot id"'] & \partial S \cdot \partial S \arrow[r, "\rho"'] & S & \partial S \cdot \partial S \arrow[l, "\rho"'] & \partial(S^{\bullet}) \cdot \partial S \arrow[l, "\partial\pi_1 \cdot id"']
\end{tikzcd}$$

Diagram 1

We get the sequence

$$(f, g) \mapsto ((\{*_\emptyset\}, f), g) \mapsto ((\{*_\emptyset, *_{*_\emptyset}\}, f), g) \mapsto ((\{*_\emptyset, *_{X_1}\}, f), g) \mapsto ((f, *_{X_1}), g) \mapsto (f, g)\,.$$

Hence, (3.3.2) is the identity on $\partial S{\cdot}\partial S$. By the equalities behind the commutations of D_l, D_b, D_m, D_t and D_r, we get the sequence of equalities

$$\begin{aligned}
\rho &= \rho \circ id \\
&= \rho \circ ((\partial\pi_1 \circ \partial\lambda^{\blacktriangle} \circ \varphi^{-1} \circ i_l \circ (\partial\epsilon_2^{-1} \cdot id) \circ \lambda^{\star -1}) \cdot id) \\
&= \rho \circ ((\partial\rho \circ \varphi^{-1} \circ i_l \circ (\partial\partial\eta_2 \cdot id) \circ (\partial\epsilon_2^{-1} \cdot id) \circ \lambda^{\star -1}) \cdot id)
\end{aligned}$$

$$
\begin{aligned}
&= \rho\circ((\partial\rho\circ\varphi^{-1}\circ i_l)\cdot id)\circ\theta_1\circ(((\partial\partial\eta_2\cdot id)\circ(\partial\epsilon_2^{-1}\cdot id)\circ\lambda^{\star-1})\cdot id)\\
&= \rho\circ((\partial\rho\circ\varphi^{-1}\circ i_l\circ(\partial\partial\eta_2\cdot id))\cdot id)\circ\theta_1\circ(((\partial\epsilon_2^{-1}\cdot id)\circ\lambda^{\star-1})\cdot id)\\
&= \rho\circ((\partial\rho\circ\varphi^{-1}\circ i_l\circ(\partial\partial\eta_2\cdot id)\circ(\partial\epsilon_2^{-1}\cdot id)\circ\lambda^{\star-1})\cdot id)\circ\gamma\\
&= \rho\circ((\partial\pi_1\circ\partial\lambda^{\blacktriangle}\circ\varphi^{-1}\circ i_l\circ(\partial\epsilon_2^{-1}\cdot id)\circ\lambda^{\star-1})\cdot id)\circ\gamma\\
&= \rho\circ\gamma\,,
\end{aligned}
$$

which proves the claim. ■

As a consequence of Lemma 3.20, the verification of the axiom (CA1) comes down to the verification of its instance $\rho_{21}\circ\theta_1=\rho_{11}$.

Corollary 3.21. *The equality $\rho_2\circ\theta=\rho_1$ holds if and only if the equality $\rho_{21}\circ\theta_1=\rho_{11}$ holds.*

Together with the equality $\kappa_S^{\blacktriangle}=\lambda_S^{\blacktriangle}\circ\gamma_{\partial S,\partial E_2}$, Lemma 3.20 is also used to prove the algebraic analogue of the right unitality law (U2).

Corollary 3.22. *The morphism $\eta_2 : E_2\rightarrow S$ satisfies the coherence condition given by the commutation of the diagram*

$$
\begin{tikzcd}
S\blacktriangle S \arrow[d, "\rho"'] & S\blacktriangle E_2 \arrow[l, "id\blacktriangle\eta_2"'] \arrow[d, "\kappa_S^{\blacktriangle}"] \\
S & S^{\bullet} \arrow[l, "\pi_{1S}"]
\end{tikzcd}
$$

The following theorem ensures that Definition 3.19 does the job. In order to make its statement concise (as well as the statements of Theorem 3.30 and Theorem 3.35 later), we adopt the following convention.

Convention 3.23. *We say that two definitions are* equivalent *if, given a structure specified by the first definition, one can construct a structure specified by the second definition, and vice-versa, in such a way that going from one structure to the other one, and back, leads to a structure isomorphic to the starting one. If the latter transformations results exactly in the initial structure, we say that the corresponding definitions are* strongly equivalent*. Categorically speaking, the equivalence and the strong equivalence of definitions are nothing but the equivalence and the isomorphism of the categories of appropriate structures, respectively.*

Theorem 3.24. *Definition 1.4 (entries-only, biased) and Definition 3.19 (entries-only, algebraic) are strongly equivalent.*

Proof. We define functors in both directions and show that going from one structure to the other one, and back, leads to the same structure.

Biased to Algebraic. Let $\mathcal{C}:\mathbf{Bij}^{op}\rightarrow\mathbf{Set}$ be an entries-only cyclic operad defined in biased manner. The algebraic cyclic operad structure over the species $\mathcal{C}$ is derived as follows. For a finite set X, a decomposition (X_1,X_2) of X, $f\in\partial\mathcal{C}(X_1)$ and $g\in\partial\mathcal{C}(X_2)$, $\rho_X:(\partial\mathcal{C}\cdot\partial\mathcal{C})(X)\rightarrow\mathcal{C}(X)$ is defined by setting

$$\rho_X(f,g)=f\;{}_{*_{X_1}}\!\circ_{*_{X_2}}\;g\,.$$

For a two-element set, say $\{x,y\}$, the morphism $\eta:E_2\rightarrow\mathcal{C}$ is defined as $\eta_{\{x,y\}}:\{x,y\}\mapsto id_{x,y}$. Otherwise, η_X is the empty function. We now verify the axioms.

(CA1) We prove the equality $\rho_{21}\circ\theta_1=\rho_{11}$ by chasing Diagram 2, obtained by unfolding the definitions of the three morphisms involved. The axiom (CA1) then follows by Corollary 3.21.

$$
\begin{tikzcd}
(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[r, "\alpha"] \arrow[d, "(\varphi^{-1}\circ i_l)\cdot id"'] & \partial\partial S\cdot(\partial S\cdot\partial S) \arrow[r, "\mathtt{ex}\cdot\gamma"] & \partial\partial S\cdot(\partial S\cdot\partial S) \arrow[r, "\alpha^{-1}"] & (\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "(\varphi^{-1}\circ i_l)\cdot id"] \\
\partial(\partial S\cdot\partial S)\cdot\partial S \arrow[r, "\partial\rho\cdot id"'] & \partial S\cdot\partial S \arrow[r, "\rho"'] & S & \partial S\cdot\partial S \arrow[l, "\rho"] & \partial(\partial S\cdot\partial S)\cdot\partial S \arrow[l, "\partial\rho\cdot id"]
\end{tikzcd}
$$

Diagram 2

Let $((f,g),h) \in ((\partial\partial S \cdot \partial S)\cdot \partial S)(X)$, and suppose that

$$f \in \partial\partial S(X_1'),\;\; g \in \partial S(X_1''),\;\; h \in \partial S(X_2),\;\; (f,g) \in (\partial\partial S \cdot \partial S)(X_1),$$

where (X_1', X_1'') is a decomposition of X_1 and (X_1, X_2) is a decomposition of X.

By chasing the diagram to the down-right starting from $((f,g),h)$, we traverse the following sequence of elements:

$$\begin{array}{rcl} ((f,g),h) & \mapsto & ((f^{\tau_1^+},g),h) \\ & \mapsto & (f^{\tau_1^+} {}_{*_{X_1'\cup\{*_{X_1}\}}}\circ_{*_{X_1''}} g, h) \\ & \mapsto & (f^{\tau_1^+} {}_{*_{X_1'\cup\{*_{X_1}\}}}\circ_{*_{X_1''}} g) {}_{*_{X_1}}\circ_{*_{X_2}} h\,. \end{array}$$

The first step here corresponds to the application of $(\varphi^{-1}\circ i_l)\cdot id$ on $((f,g),h)$, and, therefore, it involves the renaming $\tau_1 : X_1' \cup \{*_{X_1}\} \to X_1' \cup \{*_{X_1'}\}$ of $*_{X_1'}$ to $*_{X_1}$, i.e. the action of $\partial S(\tau_1) = S(\tau_1^+)$ on

$$f \in \partial S(X_1' \cup \{*_{X_1'}\}) = S(X_1' \cup \{*_{X_1'}\} \cup \{*_{X_1'\cup\{*_{X_1'}\}}\}),$$

where τ_1^+ is the ∂-extension of τ_1 (see Definition 3.8). Therefore,

$$f^{\tau_1^+} \in \partial S(X_1' \cup \{*_{X_1}\}) = S(X_1' \cup \{*_{X_1}\} \cup \{*_{X_1'\cup\{*_{X_1}\}}\})\,,$$

and

$$(f^{\tau_1^+}, g) \in \partial(\partial S \cdot \partial S)(X_1) = (\partial S \cdot \partial S)(X_1 \cup \{*_{X_1}\}).$$

The action of $\partial\rho$ on $(f^{\tau_1^+}, g)$ composes $f^{\tau_1^+}$ and g along the corresponding distinguished entries $*_{X_1'\cup\{*_{X_1}\}}$ and $*_{X_1''}$ (while carrying over the distinguished element $*_{X_1}$ from the pair to the composite of its components), and, finally, the action of ρ on $(f^{\tau_1^+} {}_{*_{X_1'\cup\{*_{X_1}\}}}\circ_{*_{X_1''}} g, h)$ results in the partial composition of the two components along $*_{X_1}$ and $*_{X_2}$.

The sequence on the right-down-left side consists of the sequence

$$((f,g),h) \;\;\mapsto\;\; (f,(g,h)) \;\;\mapsto\;\; (f^{\varepsilon},(h,g)) \;\;\mapsto\;\; ((f^{\varepsilon},h),g),$$

arising from the action of $\theta_1 = \alpha^{-1}\circ(\mathtt{ex}\cdot\gamma)\circ\alpha$, where

$$\varepsilon : X_1' \cup \{*_{X_1'}, *_{X_1'\cup\{*_{X_1'}\}}\} \to X_1' \cup \{*_{X_1'}, *_{X_1'\cup\{*_{X_1'}\}}\}$$

exchanges $*_{X_1'}$ and $*_{X_1'\cup\{*_{X_1'}\}}$, followed by the sequence

$$\begin{array}{rcl} ((f^{\varepsilon},h),g) & \mapsto & (((f^{\varepsilon})^{\tau_2^+},h),g) \\ & \mapsto & (((f^{\varepsilon})^{\tau_2^+} {}_{*_{X_1'\cup\{*_{X_1'\cup X_2}\}}}\circ_{*_{X_2}} h),g) \\ & \mapsto & ((f^{\varepsilon})^{\tau_2^+} {}_{*_{X_1'\cup\{*_{X_1'\cup X_2}\}}}\circ_{*_{X_2}} h) {}_{*_{X_1'\cup X_2}}\circ_{*_{X_1''}} g\,, \end{array}$$

corresponding to the action of ρ_{21}. Similarly as before, the action of $(\varphi^{-1}\circ i_l)\cdot id$ on $((f^{\varepsilon},h),g)$ involves the renaming $\tau_2 : X_1' \cup \{*_{X_1'\cup X_2}\} \to X_1' \cup \{*_{X_1'}\}$ of $*_{X_1'}$ to $*_{X_1'\cup X_2}$, i.e. the action of $\partial S(\tau_2) = S(\tau_2^+)$ on

$$f^{\varepsilon} \in \partial S(X_1' \cup \{*_{X_1'}\}) = S(X_1' \cup \{*_{X_1'}\} \cup \{*_{X_1'\cup\{*_{X_1'}\}}\}),$$

where τ_2^+ is the ∂-extension of τ_2. This results in

$$(f^{\varepsilon})^{\tau_2^+} \in \partial S(X_1' \cup \{*_{X_1'\cup X_2}\}) = S(X_1' \cup \{*_{X_1'\cup X_2}\} \cup \{*_{X_1'\cup\{*_{X_1'\cup X_2}\}}\}),$$

i.e.

$$((f^{\varepsilon})^{\tau_2^+}, h) \in \partial(\partial S \cdot \partial S)(X_1' \cup X_2) = (\partial S \cdot \partial S)(X_1' \cup X_2 \cup \{*_{X_1' \cup X_2}\}).$$

The application of $\partial\rho$ on $((f^{\varepsilon})^{\tau_2^+}, h)$ composes $(f^{\varepsilon})^{\tau_2^+}$ and h along $*_{X_1' \cup \{*_{X_1' \cup X_2}\}}$ and $*_{X_2}$ (and carries over the distinguished element $*_{X_1' \cup X_2}$ to the composition). Finally, the application of ρ on $((f^{\varepsilon})^{\tau_2^+} {}_{*_{X_1' \cup \{*_{X_1' \cup X_2}\}}}\circ_{*_{X_2}} h, g)$ composes the two components along $*_{X_1' \cup X_2}$ and $*_{X_1''}$.

The axiom (CA1) follows thanks to the axioms (A2) and (EQ) of the biased structure, which prove the sequence of equalities

$$\begin{aligned}
(f^{\tau_1^+} {}_{*_{X_1' \cup \{*_{X_1}\}}}\circ_{*_{X_1''}} g) {}_{*_{X_1}}\circ_{*_{X_2}} h &= (f^{\tau_1^+} {}_{*_{X_1}}\circ_{*_{X_2}} h) {}_{*_{X_1' \cup \{*_{X_1}\}}}\circ_{*_{X_1''}} g \\
&= (((f^{\tau_1^+})^{\sigma} {}_{*_{X_1' \cup \{*_{X_1' \cup X_2}\}}}\circ_{*_{X_2}} h) {}_{*_{X_1' \cup X_2}}\circ_{*_{X_1''}} g \\
&= ((f^{\varepsilon})^{\tau_2^+} {}_{*_{X_1' \cup \{*_{X_1' \cup X_2}\}}}\circ_{*_{X_2}} h) {}_{*_{X_1' \cup X_2}}\circ_{*_{X_1''}} g,
\end{aligned}$$

where $\sigma : X_1' \cup \{*_{X_1' \cup X_2}, *_{X_1' \cup \{*_{X_1' \cup X_2}\}}\} \to X_1' \cup \{*_{X_1}, *_{X_1' \cup \{*_{X_1}\}}\}$ renames $*_{X_1}$ to $*_{X_1' \cup \{*_{X_1' \cup X_2}\}}$ and $*_{X_1' \cup \{*_{X_1}\}}$ to $*_{X_1' \cup X_2}$. The last equality in the sequence holds by the equality $\tau_1^+ \circ \sigma = \tau_2^+ \circ \varepsilon$.

(CA2) The commutation of the diagram

$$\begin{tikzcd}
(E_2 \blacktriangle S)(X) \arrow[r, "(\eta \blacktriangle id)_X"] \arrow[d, "\lambda^{\blacktriangle}_{S X}"'] & (S \blacktriangle S)(X) \arrow[d, "\rho_X"] \\
S^{\bullet}(X) \arrow[r, "\pi_{1 S X}"'] & S(X)
\end{tikzcd}$$

for $X = \emptyset$ follows since $(E_2 \blacktriangle S)(\emptyset) = \emptyset$ and since there is a unique empty function with codomain $S(X)$. If $X \neq \emptyset$, then for $(\{x, *_{\{x\}}\}, f) \in (E_2 \blacktriangle S)(X)$ (see (3.3.1)), by chasing the down-right side of the diagram, we get

$$(\{x, *_{\{x\}}\}, f) \mapsto (f^{\sigma}, x) \mapsto f^{\sigma},$$

where σ renames $*_{X \backslash \{x\}}$ to x. By going to the right-down, we get

$$(\{x, *_{\{x\}}\}, f) \mapsto (id_{x, *_{\{x\}}}, f) \mapsto id_{x, *_{\{x\}}} {}_{*_{\{x\}}}\circ_{*_{X \backslash \{x\}}} f\,.$$

The equality $f^{\sigma} = id_{x, *_{\{x\}}} {}_{*_{\{x\}}}\circ_{*_{X \backslash \{x\}}} f$ follows easily by (U1) and (EQ).

Algebraic to Biased. Suppose that $S : \mathbf{Bij}^{op} \to \mathbf{Set}$ is an algebraic entries-only cyclic operad, let X and Y be non-empty finite sets, let $x \in X$ and $y \in Y$ be such that $X \backslash \{x\} \cap Y \backslash \{y\} = \emptyset$, and let $f \in S(X)$ and $g \in S(Y)$. Then $(f, x) \in S^{\bullet}(X)$ and $(g, y) \in S^{\bullet}(Y)$. Therefore, for

$$(\{x\}, f^{\sigma_1}) = \zeta(f, x) \in E_1 \cdot \partial S(X) \quad \text{and} \quad (\{y\}, g^{\sigma_2}) = \zeta(g, y) \in E_1 \cdot \partial S(Y),$$

where ζ is the isomorphism from Remark 3.18, we have that $f^{\sigma_1} \in \partial S(X \backslash \{x\})$ and $g^{\sigma_2} \in \partial S(Y \backslash \{y\})$. We define the partial composition ${}_x\circ_y : S(X) \times S(Y) \to S(X \backslash \{x\} \cup Y \backslash \{y\})$ as

$$f \,{}_x\circ_y\, g = \rho(f^{\sigma_1}, g^{\sigma_2}).$$

For a two-element set, say $\{x, y\}$, the distinguished element $id_{x,y} \in S(\{x, y\})$ is $\eta_{\{x,y\}}(\{x, y\})$. We move on to the verification of the axioms.

(A2) Let f and g be as above, and let $u \in X \backslash \{x\}$, $h \in S(Z)$ and $z \in Z$. We use the naturality of ρ and the commutation of Diagram 2 to prove the equality $(f \,{}_x\circ_y\, g) \,{}_u\circ_z\, h = (f \,{}_u\circ_z\, h) \,{}_x\circ_y\, g$. Since it is not evident by which element we should start the diagram chasing in order to reach $(f \,{}_x\circ_y\, g) \,{}_u\circ_z\, h$, we shall first express this composition via the multiplication ρ, and then reshape the expression we obtained towards an equal one, "accepted" by the diagram.

Firstly, we have that

$$(f\,{}_x\circ_y\, g, u) = (\rho(f^{\sigma_1}, g^{\sigma_2}), u) \in S^{\bullet}(X\backslash\{x\} \cup Y\backslash\{y\}) \quad \text{and} \quad (h,z) \in S^{\bullet}(Z).$$

By applying the isomorphism ζ from Remark 3.18 on these two elements, we get

$$(\{u\}, (f\,{}_x\circ_y\, g)^{\kappa_1}) = \zeta(f\,{}_x\circ_y\, g, u) \quad \text{and} \quad (\{z\}, h^{\kappa_2}) = \zeta(h,z),$$

where $\kappa_1 : X\backslash\{x,u\} \cup Y\backslash\{y\} \cup \{*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}\} \to X\backslash\{x\} \cup Y\backslash\{y\}$ renames u to $*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}$ and $\kappa_2 : Z\backslash\{z\} \cup \{*_{Z\backslash\{z\}}\} \to Z$ renames z to $*_{Z\backslash\{z\}}$. Therefore,

$$(f\,{}_x\circ_y\, g)^{\kappa_1} \in \partial S(X\backslash\{x,u\} \cup Y\backslash\{y\}) \quad \text{and} \quad h^{\kappa_2} \in \partial S(Z\backslash\{z\}),$$

and the left hand side of the equality (A2) becomes

$$(f\,{}_x\circ_y\, g)\,{}_u\circ_z\, h = \rho(\rho(f^{\sigma_1}, g^{\sigma_2})^{\kappa_1}, h^{\kappa_2}).$$

Next, notice that the shape of

$$\rho(f^{\sigma_1}, g^{\sigma_2})^{\kappa_1} \in S(X\backslash\{x,u\} \cup Y\backslash\{y\} \cup \{*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}\})$$

makes the element $\rho(\rho(f^{\sigma_1}, g^{\sigma_2})^{\kappa_1}, h^{\kappa_2})$ not explicitely reachable in Diagram 2. However, $\rho(f^{\sigma_1}, g^{\sigma_2})^{\kappa_1}$ is the result of chasing to the down-left the diagram below

$$\begin{tikzcd}
(S\blacktriangle S)(X\backslash\{x\} \cup Y\backslash\{y\}) \arrow[r, "(S\blacktriangle S)(\kappa_1)"] \arrow[d, "\rho_{X\backslash\{x\}\cup Y\backslash\{y\}}"'] & (S\blacktriangle S)(X\backslash\{x\} \cup Y'\backslash\{y\}) \arrow[d, "\rho_{X'\backslash\{x\}\cup Y\backslash\{y\}}"] \\
S(X'\backslash\{x\} \cup Y\backslash\{y\}) \arrow[r, "S(\kappa_1)"'] & S(X'\backslash\{x\} \cup Y\backslash\{y\})
\end{tikzcd}$$

where $X' = X\backslash\{u\} \cup \{*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}\}$, starting with $(f^{\sigma_1}, g^{\sigma_2})$. This diagram commutes as an instance of the naturality of ρ. Let us chase it to the right. Firstly, we have that

$$(S\blacktriangle S)(\kappa_1)(f^{\sigma_1}, g^{\sigma_2}) = ((\partial S)(\nu_1)(f^{\sigma_1}), (\partial S)(\nu_2)(g^{\sigma_2})) = (S(\nu_1^+)(f^{\sigma_1}), S(\nu_2^+)(g^{\sigma_2}))\,,$$

where ν_1^+ and ν_2^+ are the ∂-extensions of $\nu_1 : X\backslash\{x,u\} \cup \{*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}\} \to X\backslash\{x\}$, which renames u to $*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}$, and the identity $\nu_2 : Y\backslash\{y\} \to Y\backslash\{y\}$, respectively. Therefore, the result of chasing the diagram on the right is $\rho((f^{\sigma_1})^{\nu_1^+}, g^{\sigma_2})$. Consequently, we have that

$$(f\,{}_x\circ_y\, g)\,{}_u\circ_z\, h = \rho(\rho((f^{\sigma_1})^{\nu_1^+}, g^{\sigma_2}), h^{\kappa_2}).$$

On the other hand, chasing Diagram 2 in order to reach $(f\,{}_x\circ_y\, g)\,{}_u\circ_z\, h$ will certainly include considering the element $(\{u\}, (f^{\sigma_1})^{\alpha^+}) = \varsigma(f^{\sigma_1}, u)$, where α^+ is the ∂-extension of $\alpha : X\backslash\{x,u\} \cup \{*_{X\backslash\{x,u\}}\} \to X\backslash\{x\}$, which renames u to $*_{X\backslash\{x,u\}}$. Therefore,

$$(f^{\sigma_1})^{\alpha^+} \in \partial\partial S(X\backslash\{x,u\}),$$

and, consequently,

$$((f^{\sigma_1})^{\alpha^+}, g^{\sigma_2}) \in (\partial S \cdot \partial\partial S)(X\backslash\{x,u\} \cup Y\backslash\{y\}).$$

Furthermore, this chasing will include the element

$$\varphi^{-1}((f^{\sigma_1})^{\alpha^+}, g^{\sigma_2}) = (((f^{\sigma_1})^{\alpha^+})^{\tau^+}, g^{\sigma_2}) \in \partial(\partial S \cdot \partial S),$$

where τ^+ is the ∂-extension of $\tau : X\backslash\{x,u\}\cup\{*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}\} \to X\backslash\{x,u\}\cup\{*_{X\backslash\{x,u\}}\}$, which renames $*_{X\backslash\{x,u\}}$ to $*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}$.

As a consequence of the equality $\nu_1^+ = \alpha^+ \circ \tau^+$, we get the equality

$$(f\,{}_x\circ_y\, g)\,{}_u\circ_z\, h = \rho(\rho((((f^{\sigma_1})^{\alpha^+})^{\tau^+}, g^{\sigma_2}), h^{\kappa_2}),$$

in which the right hand side is the result of chasing Diagram 2 to the down-left, starting with

$$(((f^{\sigma_1})^{\alpha^+}, g^{\sigma_2}), h^{\kappa_2}) \in ((\partial\partial S \cdot \partial S)\cdot \partial S)(X\backslash\{x,u\}\cup Y\backslash\{y\}\cup Z\backslash\{z\}).$$

The remaining of the proof of (A2) now unfolds easily: the sequence obtained by chasing Diagram 2 to the right-down, starting with $(((f^{\sigma_1})^{\alpha^+}, g^{\sigma_2}), h^{\kappa_2})$, consists of

$$\begin{array}{rcl} (((f^{\sigma_1})^{\alpha^+}, g^{\sigma_2}), h^{\kappa_2}) & \mapsto & ((f^{\sigma_1})^{\alpha^+}, (g^{\sigma_2}, h^{\kappa_2}))\\ & \mapsto & (((f^{\sigma_1})^{\alpha^+})^{\varepsilon}, (h^{\kappa_2}, g^{\sigma_2}))\\ & \mapsto & ((((f^{\sigma_1})^{\alpha^+})^{\varepsilon}, h^{\kappa_2}), g^{\sigma_2}), \end{array}$$

arising from the action of $\theta_1 = \alpha^{-1}\circ(\mathtt{ex}\cdot\gamma)\circ\alpha$, where

$$\varepsilon : X\backslash\{x,u\}\cup\{*_{X\backslash\{x,u\}}, *_{X\backslash\{x,u\}\cup\{*_{X\backslash\{x,u\}}\}}\} \to X\backslash\{y,u\}\cup\{*_{X\backslash\{x,u\}}, *_{X\backslash\{x,u\}\cup\{*_{X\backslash\{x,u\}}\}}\}$$

exchanges $*_{X\backslash\{x,u\}}$ and $*_{X\backslash\{x,u\}\cup\{*_{X\backslash\{x,u\}}\}}$, followed by

$$\begin{array}{rcl} ((((f^{\sigma_1})^{\alpha^+})^{\varepsilon}, h^{\kappa_2}), g^{\sigma_2}) & \mapsto & (((((f^{\sigma_1})^{\alpha^+})^{\varepsilon})^{\omega^+}, h^{\kappa_2}), g^{\sigma_2})\\ & \mapsto & (\rho((((f^{\sigma_1})^{\alpha^+})^{\varepsilon})^{\omega^+}, h^{\kappa_2}), g^{\sigma_2})\\ & \mapsto & \rho(\rho((((f^{\sigma_1})^{\alpha^+})^{\varepsilon})^{\omega^+}, h^{\kappa_2}), g^{\sigma_2}), \end{array}$$

where ω^+ is the ∂-extension of the renaming $\omega : X\backslash\{x,u\}\cup\{*_{X\backslash\{x,u\}\cup Z\backslash\{z\}}\} \to X\backslash\{x,u\}\cup\{*_{X\backslash\{x,u\}}\}$ of $*_{X\backslash\{x,u\}}$ to $*_{X\backslash\{x,u\}\cup Z\backslash\{z\}}$. Similarly as we did for the left side of (A2), it can be shown that

$$\rho(\rho((((f^{\sigma_1})^{\alpha^+})^{\varepsilon})^{\omega^+}, h^{\kappa_2}), g^{\sigma_2}) = f\,{}_x\circ_y\,(g\,{}_u\circ_z\, h),$$

which completes the proof of (A2).

(EQ) Let f and g be as above and suppose that σ_1,σ_2 and σ are as in (EQ). Then

$$(f^{\sigma_1}, \sigma_1^{-1}(x)) \in S^\bullet(X') \quad \text{and} \quad (g^{\sigma_2}, \sigma_2^{-1}(y)) \in S^\bullet(Y'),$$

and we have

$$f^{\sigma_1}\;{}_{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)}\; g^{\sigma_2} = \rho((f^{\sigma_1})^{\tau_1}, (g^{\sigma_2})^{\tau_2}),$$

where τ_1 renames $\sigma_1^{-1}(x)$ to $*_{X'\backslash\{\sigma_1^{-1}(x)\}}$ and τ_2 renames $\sigma_2^{-1}(y)$ to $*_{Y'\backslash\{\sigma_2^{-1}(y)\}}$. On the other hand, we have that

$$(f\,{}_x\circ_y\, g)^\sigma = \rho(f^{\kappa_1}, g^{\kappa_2})^\sigma,$$

where κ_1 renames x to $*_{X\backslash\{x\}}$ and κ_2 renames y to $*_{Y\backslash\{y\}}$. The equality $\rho((f^{\sigma_1})^{\tau_1}, (g^{\sigma_2})^{\tau_2}) = \rho(f^{\kappa_1}, g^{\kappa_2})^\sigma$ follows easily by chasing the diagram

$$\begin{array}{ccc} (S\blacktriangle S)(X\backslash\{x\}\cup Y\backslash\{y\}) & \xrightarrow{(S\blacktriangle S)(\sigma)} & (S\blacktriangle S)(X'\backslash\{\nu_1^{-1}(x)\}\cup Y'\backslash\{\nu_2^{-1}(y)\})\\ \Big\downarrow{\scriptstyle \rho_{X\backslash\{x\}\cup Y\backslash\{y\}}} & & \Big\downarrow{\scriptstyle \rho_{X'\backslash\{\nu_1^{-1}(x)\}\cup Y'\backslash\{\nu_2^{-1}(y)\}}}\\ S(X\backslash\{x\}\cup Y\backslash\{y\}) & \xrightarrow[S(\sigma)]{} & S(X'\backslash\{\nu_1^{-1}(x)\}\cup Y'\backslash\{\nu_2^{-1}(y)\}) \end{array}$$

which is an instance of the naturality of ρ, starting with $(f^{\kappa_1}, g^{\kappa_2})$.

`(U1)` For $f \in S(X)$ and $x, y \in X$ we have

$$id_{x,y\ y}\circ_x f = \rho(id_{x,y}^{\tau_1}, f^{\tau_2}) = \rho(id_{*_{\{z\}},z}, f^{\tau_2}),$$

where τ_1 renames y to $*_{\{x\}}$ and τ_2 renames x to $*_{X\backslash\{x\}}$. The right hand side of the previous equality is the result of chasing to the right-down the diagram

$$\begin{tikzcd}
(E_2 \blacktriangle S)(X) \arrow[r, "(\eta_2 \blacktriangle id)_X"] \arrow[d, "\lambda^{\blacktriangle}_{S\,X}"'] & (S \blacktriangle S)(X) \arrow[d, "\rho_X"] \\
S^{\bullet}(X) \arrow[r, "\pi_{1S\,X}"'] & S(X)
\end{tikzcd}$$

which commutes by `(CA2)`, starting with $(\{x, *_{\{x\}}\}, f^{\tau_2})$. By chasing it to the down-right we get exactly f, which completes the proof of `(U1)` .

`(UP)` The preservation of units follows directly by the naturality of η_2.

Both transitions clearly leave the underlying functor unchanged and preserve units. As for the transition of the partial compostion operation, we have

$$f\ {}_x\circ_y\, g \quad\longmapsto\quad \rho(f^{\sigma_1}, g^{\sigma_2}) \quad\longmapsto\quad f^{\sigma_1}\ {}_{*_{X\backslash\{x\}}}\circ_{*_{Y\backslash\{y\}}}\ g^{\sigma_2}$$

and the conclusion follows by `(EQ)`, while, for the transition of the multiplication, we have

$$\rho_X(f, g) \quad\longmapsto\quad f\ {}_{*_{X_1}}\circ_{*_{X_2}}\, g \quad\longmapsto\quad \rho_X(f^{\tau_1}, g^{\tau_2}),$$

where τ_1 and τ_2 are identities. This completes the proof. ■

Reflecting once more the microcosm principle philosophy, given a category $\mathbf{C}$ equipped with a bifunctor $\diamond : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ that does not bear a monoidal structure, a question of finding the "minimal" associativity-like and unit-like isomorphisms can be asked, which leads to categorifications of monoid-like algebraic structures in a way analogous to the one illustrated in Table 1 in Introduction. If such isomorphisms (and unit-like objects) are established, we could say that $\mathbf{C}$ is a monoidal-*like* category and define in a natural way a monoid-like object in $\mathbf{C}$. In this section we exhibited one such monoidal-like category: $(\mathbf{Spec}, \blacktriangle, E_2)$, which, thanks to Theorem 3.24, allowes us to give the following algebraic description of entries-only cyclic operads:

A cyclic operad is a monoid-like object in the monoidal-like category $(\mathbf{Spec}, \blacktriangle, E_2)$,

the exact meaning of which can be read from Table 6.

	MONOIDAL-LIKE CATEGORY $\mathbf{Spec}$	MONOID-LIKE OBJECT $S \in \mathbf{Spec}$
PRODUCT	$\blacktriangle : \mathbf{Spec} \times \mathbf{Spec} \to \mathbf{Spec}$	$\rho : S \blacktriangle S \to S$
UNIT	$E_2 \in \mathbf{Spec}$	$\eta_2 : E_2 \to S$
ASSOCIATIVITY[2]	$\gamma_{S,T,U} : (S\blacktriangle T)\blacktriangle U + T\blacktriangle(S\blacktriangle U) + (T\blacktriangle U)\blacktriangle S$ $\to S\blacktriangle(T\blacktriangle U) + (S\blacktriangle U)\blacktriangle T + U\blacktriangle(S\blacktriangle T)$	`(CA1)`
LEFT UNIT[3]	$\lambda^{\blacktriangle}_S : E_2 \blacktriangle S \to S^{\bullet}$	`(CA2)`
RIGHT UNIT[3]	$\kappa^{\blacktriangle}_S : S \blacktriangle E_2 \to S^{\bullet}$	Corolary 3.22

TABLE 6: A cyclic operad defined internally to the monoidal-like category of species

[2]Actually, the "minimal" associativity-like isomorphism.
[3]Analogously, the "minimal" unit-like isomorphism.

3.3.2 Exchangeable-output

In this part, we first transfer Markl's skeletal exchangeable-output definition [Mar08, Proposition 42] to the non-skeletal setting, by introducing a non-skeletal version of the cycle $\tau_n = (0, 1, \ldots, n)$ that enriches the operad structure to the structure of cyclic operads. We then deliver the algebraic counterpart of the obtained non-skeletal definition.

Non-skeletal biased exchangeable-output definition

The symmetric group $\mathbb{S}_n$, whose action (in the skeletal operad structure) formalizes the permutations of the inputs of an n-ary operation, together with the action of τ_n, generates all possible permutations of the set $\{0, 1, \ldots, n\}$. Hence, they constitute the action of $\mathbb{S}_{n+1}$, which involves the action of exchanging the output of an operation (now denoted with 0) with one of the inputs. Observe that $\mathbb{S}_{n+1}$ can equivalently be generated by extending the action $\mathbb{S}_n$ with transpositions of the form $(i\ 0)$, for $1 \leq i \leq n$. In the non-skeletal setting, where the inputs of an operation are labeled with arbitrary letters, rather than with natural numbers, we mimick these transpositions with actions of the form $D_x : \mathcal{O}(X) \to \mathcal{O}(X)$, where $x \in X$ denotes the input of an operation chosen to be exchanged with the output. Here is the resulting definition.

Definition 3.25. An exchangeable-output cyclic operad is a symmetric operad $\mathcal{O}$, enriched with actions

$$D_x : \mathcal{O}(X) \to \mathcal{O}(X),$$

defined for all $x \in X$ and subject to the axioms given below, wherein, for each of the axioms, we assume that $f \in \mathcal{O}(X)$.

Preservation of units.

[DID] $D_x(id_x) = id_x$.

Inverse. For $x \in X$,

[DIN] $D_x(D_x(f)) = f$.

Equivariance. For $x \in X$ and an arbitrary bijection $\sigma : Y \to X$,

[DEQ] $D_x(f)^\sigma = D_{\sigma^{-1}(x)}(f^\sigma)$.

Exchange. For $x, y \in X$ and a bijection $\sigma : X \to X$ that renames x to y and y to x,

[DEX] $D_x(f)^\sigma = D_x(D_y(f))$.

Compatibility with operadic compositions. For $g \in \mathcal{O}(Y)$, the following equality holds:

[DC1] $D_y(f \circ_x g) = D_y(f) \circ_x g$, where $y \in X \backslash \{x\}$, and

[DC2] $D_y(f \circ_x g) = D_y(g)^{\sigma_1} \circ_v D_x(f)^{\sigma_2}$, where $y \in Y$, $\sigma_1 : Y\backslash\{y\} \cup \{v\} \to Y$ is a bijection that renames y to v and $\sigma_2 : X\backslash\{x\} \cup \{y\} \to X$ is a bijection that renames x to y. ☐

Notation 3.26. *For $f \in \mathcal{O}(X)$, $x \in X$ and $y \notin X\backslash\{x\}$, we write $D_{xy}(f)$ for $D_x(f)^\sigma$, where $\sigma : X\backslash\{x\} \cup \{y\} \to X$ renames x to y.*

In the following two lemmas, we present some simple properties of actions D_{xy}.

Lemma 3.27. *For $f \in \mathcal{O}(X)$ and $x \in X$, the following equalities hold:*

1. $D_{xx}(f) = D_x(f)$,
2. $D^{\mathcal{O}}_{xy}(id_x) = id_y$, *where* $y \notin X\backslash\{x\}$,
3. $D^{\mathcal{O}}_{yx}(D^{\mathcal{O}}_{xy}(f)) = f$, *where* $y \notin X\backslash\{x\}$,

4. $D^{\mathcal{O}}_{yx}(f^{\sigma}) = D^{\mathcal{O}}_{xy}(f)^{\sigma^{-1}}$, *where* $\sigma : X\backslash\{x\} \cup \{y\} \to X$ *renames* x *to* y, *and*

5. $D^{\mathcal{O}}_{yu}(f \circ_x g) = D^{\mathcal{O}}_{yu}(f) \circ_x g$, *where* $y \in X$, $g \in \mathcal{O}(Y)$ *and* $u \notin X\backslash\{y\}$.

Lemma 3.28. *For* $f \in \mathcal{O}(X)$, $x, y \in X$ *and* $z \notin X\backslash\{x, y\}$, *the following equality holds*

(DCO) $D^{\mathcal{O}}_{xy}(D^{\mathcal{O}}_{yz}(f)) = D^{\mathcal{O}}_{xz}(f)$.

Proof. Since $D^{\mathcal{O}}_{xz}(f) = D_x(f)^{\sigma} = (D_x(f)^{\sigma_1})^{\sigma_2}$, where $\sigma : X\backslash\{x\} \cup \{z\} \to X$ renames x to z, $\sigma_1 : X \to X$ exchanges x and y, and $\sigma_2 : X\backslash\{x\} \cup \{z\} \to X$ renames y to z and x to y, by [DEX], we have

$$D^{\mathcal{O}}_{xz}(f) = (D^{\mathcal{O}}_{x}(D^{\mathcal{O}}_{y}(f)))^{\sigma_2}.$$

For the left side of the above equality, we have

$$D^{\mathcal{O}}_{xy}(D^{\mathcal{O}}_{yz}(f)) = (D^{\mathcal{O}}_{x}(D^{\mathcal{O}}_{y}(f)^{\tau_1}))^{\tau_2},$$

where $\tau_1 : X\backslash\{y\} \cup \{z\} \to X$ renames y to z and $\tau_2 : X\backslash\{x\} \cup \{z\} \to X\backslash\{y\} \cup \{z\}$ renames x to y. Therefore, by [DEQ], we get that

$$D^{\mathcal{O}}_{xy}(D^{\mathcal{O}}_{yz}(f)) = (D^{\mathcal{O}}_{x}(D^{\mathcal{O}}_{y}(f))^{\tau_1})^{\tau_2} = (D^{\mathcal{O}}_{x}(D^{\mathcal{O}}_{y}(f)))^{\tau_1 \circ \tau_2}.$$

The conclusion follows from the equality $\sigma_2 = \tau_1 \circ \tau_2$. ∎

We make necessary preparations for the proof of equivalence of Definition 3.25 and Definition 1.4 (see Convention 3.23).

For an exchangeable-output cyclic operad $\mathcal{O}$ and a finite set X, we introduce an equivalence relation $\approx$ on the set $\sum_{x\in X} \mathcal{O}(X\backslash\{x\})$ of (ordered) pairs (x, f), where $x \in X$ and $f \in \mathcal{O}(X\backslash\{x\})$: $\approx$ is the reflexive closure of the familly of equalities

$$(x, f) \approx (y, D_{yx}(f)), \tag{3.3.3}$$

where $y \in X\backslash\{x\}$ is arbitrary. Observe that, by Lemma 3.27(3) and (DCO), for each $x \in X$, an equivalence class

$$[(x, f)]_{\approx} \in \sum_{x\in X} \mathcal{O}(X\backslash\{x\})/_{\approx}$$

has a unique representative of the form $(x, \_)$. In other words, if $(x, f) \approx (x, g)$, then $f = g$.

In the next remark we exhibit a property of $\approx$ that we shall also need for the proof of the equivalence.

Remark 3.29. *By Lemma 3.27(5) and* (DCO), *we have that* $(y, D_{yx}(f) \circ_x g) \approx (z, D_{zx}(f) \circ_x g)$.

Finally, here is the equivalence theorem.

Theorem 3.30. *Definition 1.4 (entries-only, biased) and Definition 3.25 (exchangeable-output, biased), restricted to constant-free cyclic operads, are equivalent.*

Proof. We define functors in both directions and show that going from one structure to the other one, and back, leads to a structure isomorphic to the initial one.

Entries-only to Exchangeable-output. Let $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$ be an entries-only cyclic operad. For a finite set X and a bijection $\sigma : Y \to X$, the species $\mathcal{O}_{\mathcal{C}} : \mathbf{Bij}^{op} \to \mathbf{Set}$, underlying the corresponding exchangeable-output cyclic operad, is defined by

$$\mathcal{O}_{\mathcal{C}}(X) = \partial\mathcal{C}(X) \quad \text{and} \quad \mathcal{O}_{\mathcal{C}}(\sigma) = \partial\mathcal{C}(\sigma) = \mathcal{C}(\sigma^{+}). \tag{3.3.4}$$

For $f \in \mathcal{O}_{\mathcal{C}}(X)$ and $g \in \mathcal{O}_{\mathcal{C}}(Y)$, the partial composition operation $\circ_x : \mathcal{O}_{\mathcal{C}}(X) \times \mathcal{O}_{\mathcal{C}}(Y) \to \mathcal{O}_{\mathcal{C}}(X\backslash\{x\} \cup Y)$ is defined by setting

$$f \circ_x g = f^{\sigma}\; {}_{x}{\circ}_{*_Y}\; g, \tag{3.3.5}$$

where $\sigma : X \cup \{*_{X\backslash\{x\}\cup Y}\} \to X \cup \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x\}\cup Y}$. The distinguished element $id_x \in \mathcal{O}_{\mathcal{C}}(\{x\})$ is defined as $id_{x,*_{\{x\}}}$. Finally, for $f \in \mathcal{O}_{\mathcal{C}}(X)$ and $x \in X \cup \{*_X\}$, the action $D_x : \mathcal{O}_{\mathcal{C}}(X) \to \mathcal{O}_{\mathcal{C}}(X)$ is defined by setting

$$D_x(f) = \mathcal{C}(\sigma)(f), \tag{3.3.6}$$

where $\sigma : X \cup \{*_X\} \to X \cup \{*_X\}$ exchanges x and $*_X$. We verify the axioms.

`[A2]` Let f and g be as above and let $y \in X$ and $h \in \mathcal{O}_{\mathcal{C}}(Z)$. Thanks to the axioms `(EQ)` and `(A2)` of $\mathcal{C}$, the sequence of equalities

$$\begin{array}{rcl} (f \circ_x g) \circ_y h & = & (f^{\sigma}\; {}_{x}{\circ}_{*_Y}\; g)^{\tau}\; {}_{y}{\circ}_{*_Z}\; h \\ & = & ((f^{\sigma})^{\tau_1}\; {}_{x}{\circ}_{*_Y}\; g^{\tau_2})\; {}_{y}{\circ}_{*_Z}\; h \\ & = & ((f^{\sigma})^{\tau_1}\; {}_{y}{\circ}_{*_Z}\; h)\; {}_{x}{\circ}_{*_Y}\; g \\ & = & (f^{\kappa}\; {}_{y}{\circ}_{*_Z}\; h)\; {}_{x}{\circ}_{*_Y}\; g \\ & = & (f \circ_y h) \circ_x g \end{array}$$

where

- $\sigma : X \cup \{*_{X\backslash\{x\}\cup Y}\} \to X \cup \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x\}\cup Y}$,
- $\tau : X\backslash\{x\}\cup Y\cup\{*_{X\backslash\{x,y\}\cup Y\cup Z}\} \to X\backslash\{x\}\cup Y\cup\{*_{X\backslash\{x\}\cup Y}\}$ renames $*_{X\backslash\{x\}\cup Y}$ to $*_{X\backslash\{x,y\}\cup Y\cup Z}$,
- $\tau_1 = \tau|^{X\backslash\{x\}\cup\{*_{X\backslash\{x\}\cup Y}\}} \cup id_{\{x\}}$,
- $\tau_2 = \tau|^{Y} = id_{Y\cup\{*_Y\}}$, and
- $\kappa : X \cup \{*_{X\backslash\{x\}\cup Z}\} \to X \cup \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x\}\cup Z}$,

verifies `[A2]` for $\mathcal{O}_{\mathcal{C}}$. The axiom `[A1]` follows similarly, by using the (derived) equality `(A1)` of $\mathcal{C}$ (see Remark 1.8).

`[EQ]` For arbitrary bijections $\sigma_1 : X' \to X$ and $\sigma_2 : Y' \to Y$, thanks to the axiom `(EQ)` of $\mathcal{C}$, as well as its variant from Lemma 1.7, we obtain the following sequence of equalities:

$$\begin{array}{rcl} f^{\sigma_1^+} \circ_{\sigma_1^{-1}(x)} g^{\sigma_2^+} & = & (f^{\sigma_1^+})^{\tau}\; {}_{\sigma_1^{+-1}(x)}{\circ}_{*_{Y'}}\; g^{\sigma_2^+} \\ & = & (f^{\sigma_1^+})^{\tau}\; {}_{\tau^{-1}(\sigma_1^{+-1}(x))}{\circ}_{*_{Y'}}\; g^{\sigma_2^+} \\ & = & (f\; {}_{x}{\circ}_{*_Y} g)^{\kappa} \\ & = & (f^{\nu}\; {}_{x}{\circ}_{*_Y}\; g)^{\sigma^+} \\ & = & (f \circ_x g)^{\sigma^+} \end{array}$$

where

- $\tau : X' \cup \{*_{X'\backslash\{\sigma_1^{-1}(x)\}\cup Y'}\} \to X' \cup \{*_{X'}\}$ renames $*_{X'}$ to $*_{X'\backslash\{\sigma_1^{-1}(x)\}\cup Y'}$,
- $\nu : X \cup \{*_{X\backslash\{x\}\cup Y}\} \to X \cup \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x\}\cup Y}$,
- $\kappa = (\sigma_1^+ \circ \tau)|^{X\backslash\{x\}\cup\{*_X\}} \cup \sigma_2^+|^{Y}$, and
- $\sigma = \sigma_1|^{X\backslash\{x\}} \cup \sigma_2$,

which proves [EQ]. Observe that

$$\kappa = (\nu|^{X\backslash\{x\}\cup\{*_X\}} \cup id_Y) \circ \sigma^+,$$

which justifies the application of Lemma 1.7 to get the equality $(f\,{}_x\circ_{*_Y} g)^\kappa = (f^\nu\,{}_x\circ_{*_Y}\, g)^{\sigma^+}$.

[U1] By the axioms (UP) and (U1) for $\mathcal{C}$, for $f \in \mathcal{O}_{\mathcal{C}}(X)$ we have

$$id_y \circ_y f = id^\sigma_{y,*_{\{y\}}}\,{}_y\circ_{*_X}\, f = id_{y,*_X}\,{}_y\circ_{*_X}\, f = f,$$

where $\sigma : \{y, *_X\} \to \{y, *_{\{y\}}\}$ renames $*_{\{y\}}$ to $*_X$.
Similarily, the axioms [U2] and [UP] for $\mathcal{O}_{\mathcal{C}}$ follow thanks to the law (U2) (see Lemma 1.6) and the axiom (UP) of $\mathcal{C}$.

Concerning the axioms of the actions D_x, [DID], [DIN], [DEQ], and [DEX] follow easily by functoriality of $\mathcal{C}$. The axioms [DC1] and [DC2] require more.

[DC1] Let $f \in \mathcal{O}_{\mathcal{C}}(X), g \in \mathcal{O}_{\mathcal{C}}(Y)$, $x \in X$ and $y \in X\backslash\{x\}$. We shall use the following bijections:

- $\sigma : X\backslash\{x\} \cup Y \cup \{*_{X\backslash\{x\}\cup Y}\} \to X\backslash\{x\} \cup Y \cup \{*_{X\backslash\{x\}\cup Y}\}$, that exchanges y and $*_{X\backslash\{x\}\cup Y}$,
- $\nu : X \cup \{*_{X\backslash\{x\}\cup Y}\} \to X \cup \{*_X\}$, that renames $*_X$ to $*_{X\backslash\{x\}\cup Y}$,
- $\sigma' : X \cup \{*_{X\backslash\{x\}\cup Y}\} \to X \cup \{*_{X\backslash\{x\}\cup Y}\}$, that exchanges y and $*_{X\backslash\{x\}\cup Y}$, and
- $\tau : X \cup \{*_X\} \to X \cup \{*_X\}$, that exchanges y and $*_X$.

Observe that

$$\tau \circ \nu = \nu \circ \sigma' \quad \text{and} \quad \sigma = \sigma'|^{X\backslash\{x\}\cup\{*_{X\backslash\{x\}\cup Y}\}} \cup id_Y.$$

Thanks to the axiom (EQ) of $\mathcal{C}$, this gives us

$$D_y(f \circ_x g) = (f^\nu\,{}_x\circ_{*_Y}\, g)^\sigma = (f^\nu)^{\sigma'}\,{}_x\circ_{*_Y}\, g = (f^\tau)^\nu\,{}_x\circ_{*_Y}\, g = D_y(f) \circ_x g\,.$$

[DC2] Let f, g and x be as in the proof of [DC1] and let $y \in Y$ instead. Let σ_1 and σ_2 be as in Definition 3.25. We shall use the following bijections:

- $\tau_1 : Y\backslash\{y\} \cup \{v, *_{Y\backslash\{y\}\cup\{v\}}\} \to Y \cup \{*_Y\}$, that renames y to $*_{Y\backslash\{y\}\cup\{v\}}$ and $*_Y$ to v,
- $\tau_2 : X\backslash\{x\} \cup \{y, *_{X\backslash\{x\}\cup\{y\}}\} \to X \cup \{*_X\}$, that renames x to $*_{X\backslash\{x\}\cup\{y\}}$ and $*_X$ to y,
- $\tau : Y\backslash\{y\} \cup \{v, *_{X\backslash\{x\}\cup Y}\} \to Y\backslash\{y\} \cup \{v, *_{Y\backslash\{y\}\cup\{v\}}\}$, that renames $*_{Y\backslash\{y\}\cup\{v\}}$ to $*_{X\backslash\{x\}\cup Y}$,
- $\kappa_1 : X\backslash\{x\} \cup \{y, *_{X\backslash\{x\}\cup\{y\}}\} \to X \cup \{*_{X\backslash\{x\}\cup Y}\}$, that renames x to $*_{X\backslash\{x\}\cup\{y\}}$ and $*_{X\backslash\{x\}\cup Y}$ to y,
- $\kappa_2 : Y\backslash\{y\} \cup \{v, *_{X\backslash\{x\}\cup Y}\} \to Y \cup \{*_Y\}$, that renames y to $*_{X\backslash\{x\}\cup Y}$ and $*_Y$ to v,
- $\sigma : X\backslash\{x\} \cup Y \cup \{*_{X\backslash\{x\}\cup Y}\} \to X\backslash\{x\} \cup Y \cup \{*_{X\backslash\{x\}\cup Y}\}$, that exchanges y and $*_{X\backslash\{x\}\cup Y}$, and
- $\nu : X \cup \{*_{X\backslash\{x\}\cup Y}\} \to X \cup \{*_X\}$, that renames $*_X$ to $*_{X\backslash\{x\}\cup Y}$.

Observe that

$$\tau_2 = \nu \circ \kappa_1, \quad \kappa_2 = \tau_1 \circ \tau \quad \text{and} \quad \sigma = \kappa_1|^{X\backslash\{x\}\cup\{*_{X\backslash\{x\}\cup Y}\}} \cup \kappa_2|^Y.$$

Thanks to the axiom (EQ) and the derived law (CO) of $\mathcal{C}$, this gives us

$$\begin{array}{rcl} D^{\mathcal{O}}_y(g)^{\sigma_1} \circ_v D^{\mathcal{O}}_x(f)^{\sigma_2} & = & g^{\tau_1} \circ_v f^{\tau_2} \\ & = & (g^{\tau_1})^\tau\,{}_v\circ_{*_{X\backslash\{x\}\cup\{y\}}}\, f^{\tau_2} \end{array}$$

$$
\begin{aligned}
&= f^{\tau_2}\,{}_{*_{X\backslash\{x\}\cup\{y\}}}\circ_v\,(g^{\tau_1})^{\tau}\\
&= (f^{\nu})^{\kappa_1}\,{}_{*_{X\backslash\{x\}\cup\{y\}}}\circ_v\,g^{\kappa_2}\\
&= (f^{\nu}\,{}_x\circ_{*_Y}\,g)^{\sigma}\\
&= D^{\mathcal{O}}_y(f\circ_x g).
\end{aligned}
$$

Exchangeable-output to Entries-only. Suppose that $\mathcal{O}:\mathbf{Bij}^{op}\to\mathbf{Set}$ is an exchangeable-output cyclic operad. The species $\mathcal{C}_{\mathcal{O}}:\mathbf{Bij}^{op}\to\mathbf{Set}$, underlying the cyclic operad in the entries-only fashion, is defined by

$$\mathcal{C}_{\mathcal{O}}(X)=\sum_{x\in X}\mathcal{O}(X\backslash\{x\})/{\approx} \tag{3.3.7}$$

(see (3.3.3)). Accordingly, for $[(x,f)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(X)$ and a bijection $\sigma:Y\to X$, we set

$$\mathcal{C}_{\mathcal{O}}(\sigma)([(x,f)]_{\approx})=[(\sigma^{-1}(x),\mathcal{O}(\sigma|^{X\backslash\{x\}})(f))]_{\approx}.$$

For $[(u,f)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(X)$ and $[(v,g)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(Y)$, the partial composition operation ${}_x\circ_y:\mathcal{C}_{\mathcal{O}}(X)\times\mathcal{C}_{\mathcal{O}}(Y)\to\mathcal{C}_{\mathcal{O}}(X\backslash\{x\}\cup Y\backslash\{y\})$ is defined as follows:

$$[(u,f)]_{\approx}\,{}_x\circ_y[(v,g)]_{\approx}=\begin{cases}[(z,D_{zx}(f)\circ_x g)]_{\approx}, & \text{if } u=x \text{ and } v=y,\\ [(z,D_{zx}(f)\circ_x D_{yv}(g))]_{\approx}, & \text{if } u=x \text{ and } v\neq y,\\ [(u,f\circ_x g)]_{\approx}, & \text{if } u\neq x \text{ and } v=y,\\ [(u,f\circ_x D_{yv}(g))]_{\approx}, & \text{if } u\neq x \text{ and } v\neq y,\end{cases} \tag{3.3.8}$$

where $z\in X\backslash\{x\}$ is arbitrary. To illustrate that ${}_x\circ_y$ is well-defined, suppose, say, that (u,f) and (v,g) are such that $u=x$ and $v\neq y$ and let $s\in X\backslash\{x\}$ and $w\in Y\backslash\{v\}$ be arbitrary. Then, if, say, $w=y$, we have

$$[(s,D_{sx}(f))]_{\approx}\;{}_x\circ_y\;[(y,D_{yv}(g))]_{\approx}=[(s,D_{sx}(f)\circ_x D_{yv}(g))]_{\approx},$$

and $(z,D_{zx}(f)\circ_x D_{yv}(g))\approx(s,D_{sx}(f)\circ_x D_{yv}(g))$ by Remark 3.29. If $w\neq y$, then

$$[(s,D_{sx}(f))]_{\approx}\;{}_x\circ_y\;[(w,D_{wv}(g))]_{\approx}=[(s,D_{sx}(f)\circ_x D_{yw}(D_{wv}(g)))]_{\approx},$$

and, by `(DC0)` and Remark 3.29, we have

$$(s,D_{sx}(f)\circ_x D_{yw}(D_{wv}(g)))=(s,D_{sx}(f)\circ_x D_{yv}(g))\approx(z,D_{zx}(f)\circ_x D_{yv}(g)).$$

From Remark 3.29 it also follows that different choices of $z\in X\backslash\{x\}$ from the first two cases in the definition of $[(u,f)]_{\approx}\,{}_x\circ_y[(v,g)]_{\approx}$ lead to the same result. In the remaining of the proof, we shall assume that (x,f) and (v,g) satisfy the conditions $u\neq x$ and $v=y$. Finally, for a two-element set, say $\{x,y\}$, the distinguished element $id_{x,y}\in\mathcal{C}_{\mathcal{O}}(\{x,y\})$ will be the equivalence class $[(x,id_y)]_{\approx}$. Notice that, by `[DID]`, we have that $(x,id_y)\approx(y,id_x)$. We check the axioms.

`(A2)` Let $[(u,f)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(X)$, $[(y,g)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(Y)$, $[(w,h)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(Z)$, $x\in X$, $y\in Y$ and $w\in Z$. We prove the instance of associativity that requires the use of `[DC2]` and `[DID]`, namely

$$([(u,f)]_{\approx}\;{}_x\circ_y\;[(y,g)]_{\approx})\;{}_u\circ_w\;[(w,h)]_{\approx}=([(u,f)]_{\approx}\;{}_u\circ_w\;[(w,h)]_{\approx})\;{}_x\circ_y\;[(y,g)]_{\approx}.$$

Since $\mathcal{O}(\emptyset)=\emptyset$ and $g\in\mathcal{O}(Y\backslash\{y\})$ (resp. $h\in\mathcal{O}(Z\backslash\{w\})$), we have that $Y\backslash\{y\}\neq\emptyset$ (resp. $Z\backslash\{w\}\neq\emptyset$). Suppose that $X\backslash\{x,u\}=\emptyset$. For the expression on the left side of the above equality we then have

$$([(u,f)]_{\approx}\;{}_x\circ_y\;[(y,g)]_{\approx})\;{}_u\circ_w\;[(w,h)]_{\approx}=[(z,D_{zu}(f\circ_x g)\circ_u h)]_{\approx},$$

where we chose $z \in Y\backslash\{y\}$. On the other hand, we have

$$([(u,f)]_{\approx}\ {}_u\circ_w\ [(w,h)]_{\approx})\ {}_x\circ_y\ [(y,g)]_{\approx} = [(v, D_{vx}(D_{xu}(f)\circ_u h)\circ_x g)]_{\approx},$$

where we chose $v \in Z\backslash\{w\}$. The associativity follows if we prove that

$$D_{vz}(D_{zu}(f\circ_x g)\circ_u h) = D_{vx}(D_{xu}(f)\circ_u h)\circ_x g.$$

For this we use `[DC2]`, followed by `[DID]`, on both sides of the equality. We get

$$D_v(h)\circ_v D_{uz}(D_{zu}(f\circ_x g)) = D_v(h)\circ_v (f\circ_x g)$$

on the left side and

$$(D_v(h)\circ_v D_{ux}(D_{xu}(f)))\circ_x g = (D_v(h)\circ_v f)\circ_x g$$

on the right side, and the conclusion follows by the axiom `[A2]` of $\mathcal{O}$. If $X\backslash\{x,u\}\neq\emptyset$ and $z\in X\backslash\{x,u\}$, the associativity follows more directly by `[DC1]`, by choosing $v=z$.

`(EQ)` Let $[(u,f)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(X)$, $[(y,g)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(Y)$, and let $\sigma_1 : X'\to X$ and $\sigma_2 : Y'\to Y$ be bijections. Suppose that $\sigma_1^{-1}(x)=x'$, $\sigma_1^{-1}(u)=u'$ and $\sigma_2^{-1}(y)=y'$. We prove that

$$\mathcal{C}_{\mathcal{O}}(\sigma_1)([(u,f)]_{\approx})\ {}_{x'}\circ_{y'}\ \mathcal{C}_{\mathcal{O}}(\sigma_2)([(y,g)]_{\approx}) = \mathcal{C}_{\mathcal{O}}(\sigma)([(u,f)]_{\approx}{}_x\circ_y\ [(y,g)]_{\approx}),$$

where $\sigma=\sigma_1|^{X\backslash\{x\}}\cup\sigma_2|^{Y\backslash\{y\}}$. Let $\sigma'=\sigma|^{X\backslash\{x,u\}\cup Y\backslash\{y\}}$. Thanks to the axiom `[EQ]` of $\mathcal{O}$, we get

$$\begin{aligned}([(u,f)]_{\approx}{}_x\circ_y\ [(y,g)]_{\approx})^{\sigma} &= [(u,f\circ_x g)]_{\approx}^{\sigma}\\ &= [(u',(f\circ_x g)^{\sigma'})]_{\approx}\\ &= [(u',f^{\tau_1}\circ_{x'} g^{\tau_2})]_{\approx}\\ &= [(u',f^{\sigma_1|^{X\backslash\{u\}}}\circ_{x'} g^{\sigma_2|^{Y\backslash\{y\}}})]\\ &= [(u',f^{\sigma_1|^{X\backslash\{u\}}})]_{\approx}\ {}_{x'}\circ_{y'}\ [(y',g^{\sigma_2|^{Y\backslash\{y\}}})]_{\approx}\\ &= [(u,f)]_{\approx}^{\sigma_1}\ {}_{x'}\circ_{y'}\ [(y,g)]_{\approx}^{\sigma_2},\end{aligned}$$

where $\tau_1=\sigma|^{X\backslash\{u\}}\cup\sigma_1|^{x}$ and $\tau_2=\sigma|^{Y\backslash\{y\}}=\sigma_2|^{Y\backslash\{y\}}$.

`(U1)` For $[(u,f)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(X)$, by `[U1]`, we have

$$[(y,id_x)]_{\approx}\ {}_y\circ_u\ [(u,f)]_{\approx} = [(x,D_{xy}(id_x)\circ_y f)]_{\approx} = [(x,id_y\circ_y f)]_{\approx} = [(x,D_{xu}(f))]_{\approx} = [(u,f)]_{\approx}.$$

`(UP)` For $[(y,id_x)]_{\approx}\in\mathcal{C}_{\mathcal{O}}(\{x,y\})$, and a renaming $\sigma:\{u,v\}\to\{x,y\}$ of x to u and y to v, thanks to `[UP]`, we have

$$\mathcal{C}_{\mathcal{O}}(\sigma)(id_{x,y}) = \mathcal{C}_{\mathcal{O}}(\sigma)([(y,id_x)]_{\approx}) = [(\sigma^{-1}(y),\mathcal{O}(\sigma|^{\{x\}})(id_x))]_{\approx} = [(v,id_u)]_{\approx} = id_{u,v}.$$

The isomorphism of cyclic operads $\mathcal{C}$ and $\mathcal{C}_{\mathcal{O}_{\mathcal{C}}}$ (and $\mathcal{O}$ and $\mathcal{O}_{\mathcal{C}_{\mathcal{O}}}$). The isomorphism-of-species part, which is the same as in the proof of the equivalence of algebraic definitions, will be formally established by Lemma 3.33, as a consequence of the categorical equivalence indicated to us by Lamarche. Nevertheless, we give now the definitions of the components $\phi_{\mathcal{C}X}:\mathcal{C}_{\mathcal{O}_{\mathcal{C}}}(X)\to\mathcal{C}(X)$ and $\psi_{\mathcal{O}X}:\mathcal{O}(X)\to\mathcal{O}_{\mathcal{C}_{\mathcal{O}}}(X)$ of the isomorphisms $\phi_{\mathcal{C}}:\mathcal{C}_{\mathcal{O}_{\mathcal{C}}}\to\mathcal{C}$ and $\psi_{\mathcal{O}}:\mathcal{O}\to\mathcal{O}_{\mathcal{C}_{\mathcal{O}}}$, respectively:

- $\phi_{\mathcal{C}X}([(u,f)]_{\approx})=f^{\kappa}$, where $\kappa:X\to X\backslash\{u\}\cup\{*_{X\backslash\{u\}}\}$ renames $*_{X\backslash\{u\}}$ to u, and,
- $\psi_{\mathcal{O}X}(f)=[(*_X,f)]_{\approx}$.

As for the corresponding partial composition translations, for $[(u,f)]_{\approx} \in \mathcal{C}_{\mathcal{O}_{\mathcal{C}}}(X)$, $[(y,g)]_{\approx} \in \mathcal{C}_{\mathcal{O}_{\mathcal{C}}}(Y)$ and $x \in X\backslash\{u\}$, we have

$$[(u,f)]_{\approx}\ {}_x\circ_y\ [(y,g)]_{\approx} = [(u, f \circ_x g)]_{\approx} = [(u, f^{\sigma}\ {}_x\circ_{*_{Y\backslash\{y\}}}\ g)]_{\approx} = [(u, f^{\sigma}\ {}_x\circ_y\ g^{\tau_2})]_{\approx},$$

where $\sigma : X\backslash\{u\} \cup \{*_{X\backslash\{u,x\}\cup Y\backslash\{y\}}\} \to X\backslash\{u\} \cup \{*_{X\backslash\{u\}}\}$ renames $*_{X\backslash\{u\}}$ to $*_{X\backslash\{u,x\}\cup Y\backslash\{y\}}$ and $\tau_2 : Y \to Y\backslash\{y\} \cup \{*_{Y\backslash\{y\}}\}$ renames $*_{Y\backslash\{y\}}$ to y. Notice that for the last equality above we use the axiom (EQ) of both $\mathcal{C}_{\mathcal{O}_{\mathcal{C}}}$ and $\mathcal{C}$. The claim follows since

$$\phi_{\mathcal{C}X}([(u, f^{\sigma}\ {}_x\circ_y\ g^{\tau_2})]_{\approx}) = (f^{\sigma}\ {}_x\circ_y\ g^{\tau_2})^{\kappa} = f^{\tau_1}\ {}_x\circ_y\ g^{\tau_2},$$

where $\kappa : X\backslash\{x\} \cup Y\backslash\{y\} \to X\backslash\{x,u\} \cup Y\backslash\{y\} \cup \{*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}\}$ renames $*_{X\backslash\{x,u\}\cup Y\backslash\{y\}}$ to u and $\tau_1 : X \to X\backslash\{u\} \cup \{*_{X\backslash\{u\}}\}$ renames u to $*_{X\backslash\{u\}}$, wherein the last equality above holds by the axiom (EQ) of $\mathcal{C}$.

For $f \in \mathcal{O}(X)$ and $g \in \mathcal{O}(Y)$, we have

$$\begin{aligned}
\psi_{\mathcal{O}X}(f) \circ_x \psi_{\mathcal{O}Y}(g) &= [(*_X, f)]_{\approx} \circ_x [(*_Y, g)]_{\approx} \\
&= [(*_X, f)]^{\sigma}\ {}_x\circ_{*_Y}\ [(*_Y, g)] \\
&= [(*_{X\backslash\{x\}\cup Y}, f)]\ {}_x\circ_{*_Y}\ [(*_Y, g)]_{\approx} \\
&= [(*_{X\backslash\{x\}\cup Y}, f \circ_x g)]_{\approx} \\
&= \psi_{\mathcal{O}X}(f \circ_x g),
\end{aligned}$$

where $\sigma : X \cup \{*_{X\backslash\{x\}\cup Y}\} \to X \cup \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x\}\cup Y}$.

For the unit elements, thanks to the axioms (UP) and [UP], respectively, we have

$$\phi_{\mathcal{C}\{x,y\}}([(x, id_{y,*_{\{y\}}})]_{\approx}) = id^{\kappa}_{y,*_{\{y\}}} = id_{x,y},$$

where $\kappa : \{x,y\} \to \{y, *_{\{y\}}\}$ renames $*_{\{y\}}$ to x, and

$$\psi_{\mathcal{O}\{x\}}(id_x) = [(*_{\{x\}}, id_x)]_{\approx} = [(x, id_{*_{\{x\}}})]_{\approx},$$

which completes the proof of the theorem. ■

Algebraic definition

If we think about the algebraic variant of Definition 3.25, it is clear that its cornerstone should be an ordinary operad, i.e. a triple (S, ν, η_1) specified by Definition 3.14, and that the goal is to enrich this structure by a natural transformation which "glues together" the actions $D_x : S(X) \to S(X)$ and encompasses the coherence conditions these actions satisfy. We give the definition below.

Definition 3.31. A cyclic operad is a quadruple (S, ν, η_1, D), such that (S, ν, η_1) is an operad, and the natural transformation $D : \partial S \to \partial S$ satisfies the following laws:

(D0) $D \circ \eta^{\partial S} = \eta^{\partial S}$,

(D1) $D \circ D = id_{\partial S}$,

(D2) $(\partial D \circ \mathtt{ex}) \circ (\partial D \circ \mathtt{ex}) \circ (\partial D \circ \mathtt{ex}) = id_{\partial\partial S}$,

as well as the coherence conditions given by the commutations of the following two diagrams:

(D3)

$$
\begin{tikzcd}
\partial(\partial S)\cdot S \arrow[r, "(\mathtt{ex}\circ\partial D\circ\mathtt{ex})\cdot id"] \arrow[d, "\nu_3"'] & \partial(\partial S)\cdot S \arrow[d, "\nu_3"] \\
\partial S \arrow[r, "D"'] & \partial S
\end{tikzcd}
$$

(D4)

$$
\begin{tikzcd}
\partial S\cdot\partial S \arrow[rr, "D\cdot D"] \arrow[d, "\nu_4"'] & & \partial S\cdot\partial S \arrow[d, "\gamma"] \\
\partial S \arrow[r, "D"'] & \partial S & \partial S\cdot\partial S \arrow[l, "\nu_4"]
\end{tikzcd}
$$

in which ν_3 and ν_4 are induced from ν as follows:

$$\nu_3 : \partial\partial S\cdot S \xrightarrow{i_l} \partial\partial S\cdot S+\partial S\cdot\partial S \xrightarrow{\varphi^{-1}} \partial(\partial S\cdot S) \xrightarrow{\partial\nu} \partial S, \quad \text{and}$$
$$\nu_4 : \partial S\cdot\partial S \xrightarrow{i_r} \partial\partial S\cdot S+\partial S\cdot\partial S \xrightarrow{\varphi^{-1}} \partial(\partial S\cdot S) \xrightarrow{\partial\nu} \partial S. \qquad \square$$

That Definition 3.31 is indeed equivalent to Definition 3.25 will follow after the proof of the equivalence between Definition 3.31 and Definition 3.19 in the next section (see Table 4). As for a direct evidence, we content ourselves by showing the correspondence between the natural transformation D and the individual actions D_x. Given $D : \partial S \to \partial S$, one defines $D_x : S(X) \to S(X)$ as

$$D_x = S(\sigma^{-1}) \circ D_{X\backslash\{x\}} \circ S(\sigma), \tag{3.3.9}$$

where $\sigma : X\backslash\{x\} \cup \{*_{X\backslash\{x\}}\} \to X$ renames x to $*_{X\backslash\{x\}}$. In the opposite direction, we define $D_X : \partial S(X) \to \partial S(X)$ via D_x as

$$D_X = D_{*_X}.$$

The correspondence between the axioms of D and the ones of D_x is given in Table 7. In particular, the axiom (D2) corresponds exactly to the law (DCO) (that holds thanks to [DEQ] and [DEX], by Lemma 3.28).

D	D_x
(D0)	[DID]
(D1)	[DIN]
(D2)	[DEQ], [DEX]
(D3)	[DC1]
(D4)	[DC2]

TABLE 7: Algebraic and biased axiomatisations of the non-skeletal input-output interchange

Remark 3.32. *Notice that the axiom* (D2) *can be read as the equality*

$$\mathtt{ex}\circ\partial D\circ\mathtt{ex} = \partial D\circ\mathtt{ex}\circ\partial D.$$

Therefore, the diagram obtained from (D3) *by replacing* $(\mathtt{ex}\circ\partial D\circ\mathtt{ex})\cdot id$ *with* $(\partial D\circ\mathtt{ex}\circ\partial D)\cdot id$ *also commutes.*

To summarise, we obtained the algebraic definition of exchangeable-output cyclic operads (Definition 3.31), by first upgrading the structure $(\mathbf{Spec}, \star)$, exhibited by Fiore, into the monoidal-like category $(\mathbf{Spec}, \star, E_1)$, and then by endowing the monoid-like objects of this category, i.e. operads (see Table 8), with a natural transformation that accounts for the "input-output interchange".

	MONOIDAL-LIKE CATEGORY $\mathbf{Spec}$	MONOID-LIKE OBJECT $S \in \mathbf{Spec}$
PRODUCT	$\star : \mathbf{Spec} \times \mathbf{Spec} \to \mathbf{Spec}$	$\nu : S \star S \to S$
UNIT	$E_1 \in \mathbf{Spec}$	$\eta_1 : E_1 \to S$
ASSOCIATIVITY[4]	$\beta_{S,T,U} : (S \star T) \star U + S \star (U \star T)$ $\to S \star (T \star U) + (S \star U) \star T$	(OA1)
LEFT UNIT	$\lambda^{\star}_S : E_1 \star S \to S$	(OA2)
RIGHT UNIT	$\rho^{\star}_S : S \star E_1 \to S^{\bullet}$	

TABLE 8: An operad defined internally to the monoidal-like category of species

3.3.3 The equivalence of algebraic definitions of cyclic operads

This section deals with the proof of the equivalence between the two algebraic definitions of cyclic operads, Definition 3.19 and Definition 3.31. Based on the equivalence between the category of species which are *empty on the empty set* and the category of *species with descent data*, which was communicated to us by Lamarche, this equivalence holds for constant-free cyclic operads, i.e. cyclic operads for which the underlying species S is such that $S(\emptyset) = S(\{x\}) = \emptyset$, for all singletons $\{x\}$ (in the entries-only characterisation), and $S(\emptyset) = \emptyset$ (in the exchangeable-output characterisation). Recall that the constant-freeness requirement was also necessary for the corresponding biased equivalence, given in Theorem 3.30. The reason will soon become clear.

Descent theory for species

The equivalence of Lamarche comes from the background of descent theory. In the case of species, one starts with the question

Can we "reconstruct" a species T, given ∂T?

Intuitively, given the morphism $\partial^+ : \mathbf{Bij}^{op} \to \mathbf{Bij}^{op}$ in $\mathbf{Cat}$, defined by $\partial^+(X) = X \cup \{*_X\}$, the idea is to recover a morphism $T : \mathbf{Bij}^{op} \to \mathbf{Set}$ from $S = \partial T$ by "descending" along ∂^+, as indicated in the following diagram:

$$\begin{tikzcd} \mathbf{Bij}^{op} \arrow[r, "\partial T"] \arrow[d, "\partial^+"'] & \mathbf{Set} \arrow[d, "id_{\mathbf{Set}}"] \\ \mathbf{Bij}^{op} \arrow[r, "T"'] & \mathbf{Set} \end{tikzcd}$$

Such a reconstruction is clearly not possible without additional data, usually referred to as the *descent data*, that compensates the loss of information caused by the action of the functor ∂ : $\mathbf{Spec} \to \mathbf{Spec}$.

Lamarche defines a descent data as a pair (S, D) of a species S and a natural transformation $D : \partial S \to \partial S$, such that $D^2 = id_{\partial S}$, and $(\partial D \circ \mathtt{ex}_S)^3 = id_{\partial\partial S}$, and he proves that

> *the assignment $\partial : \mathbf{Spec}/_{\emptyset} \to \mathbf{Spec}^+$, defined by $T \mapsto (\partial T, \mathtt{ex}_T)$, is an equivalence of categories*[5].

[4]Actually, the "minimal" associativity-like isomorphism.

[5]Lamarche proves this equivalence in a skeletal setting, by considering functors of the form $S : \Sigma^{op} \to \mathbf{Set}$. The non-skeletal version that we present is an easy adaptation of his result.

Here, $\mathbf{Spec}/_{\emptyset}$ denotes the category of species S such that $S(\emptyset) = \emptyset$ and $\mathbf{Spec}^+$ denotes the category of descent data. For an object (S, D) of $\mathbf{Spec}^+$, the inverse functor

$$\int : \mathbf{Spec}^+ \to \mathbf{Spec}/_{\emptyset}$$

is defined by

$$\int (S,D)(X) = \sum_{x\in X} S(X\backslash\{x\})/_{\approx},$$

where $\approx$ is defined as in (3.3.3), whereby the actions D_x are defined via D as in (3.3.9).

The main theorem

Let $\mathbf{Spec}/_{\emptyset,\{*\}}$ be the subcategory of $\mathbf{Spec}/_{\emptyset}$, determined by species S such that $S(\emptyset) = S(\{x\}) = \emptyset$, for all singletons $\{x\}$, and let $\mathbf{Spec}^+/_{\emptyset}$ be the subcategory of $\mathbf{Spec}^+$, determined by descent data with species S for which $S(\emptyset) = \emptyset$. The following result is a direct consequence of the equivalence of Lamarche.

Lemma 3.33. *The assignment $\partial : \mathbf{Spec}/_{\emptyset,\{*\}} \to \mathbf{Spec}/^+_{\emptyset}$, defined by $T \mapsto (\partial T, \mathtt{ex}_T)$, is an equivalence of categories.*

Let $\mathbf{CO_{en}}(\mathbf{Spec}/_{\emptyset,\{*\}})$ be the category of entries-only cyclic operads (S, ρ, η_2) such that S is an object of $\mathbf{Spec}/_{\emptyset,\{*\}}$, and let $\mathbf{CO_{ex}}(\mathbf{Spec}/^+_{\emptyset})$ be the category of exchangeable-output cyclic operads (S, ν, η_1, D) such that (S, D) is an object of $\mathbf{Spec}^+/_{\emptyset}$. In both of these categories, the (iso)morphisms are natural transformations (natural isomorphisms) between underlying species which preserve the cyclic-operad structure.

The main result of this chapter is the proof that the equivalence of Lamarche carries over, via Lemma 3.33, to an equivalence between the two algebraic definitions of cyclic operads, formally given as the categorical equivalence between the two categories introduced above.

Remark 3.34. *The reason for restricting the equivalence of Lamarche to the one of Lemma 3.33 (and, therefore, to the equivalence of constant-free cyclic operads) lies in the fact that, given a species S from $\mathbf{Spec}/_{\emptyset}$, the constraint $S(\emptyset) = \emptyset$ makes the component $\rho_\emptyset : (\partial S \cdot \partial S)(\emptyset) \to S(\emptyset)$ of the multiplication $\rho : \partial S \cdot \partial S \to S$ the empty function, in which case the condition $S(\{*_\emptyset\}) = \emptyset$ is needed in order for the domain of $\rho_\emptyset$ to also be the empty set. Therefore, in the context of cyclic operads, we have to consider $\mathbf{Spec}/_{\emptyset,\{*\}}$ instead of $\mathbf{Spec}/_{\emptyset}$, and, consequently, $\mathbf{Spec}/^+_{\emptyset}$ instead of $\mathbf{Spec}^+$.*

Theorem 3.35. *The categories $\mathbf{CO_{en}}(\mathbf{Spec}/_{\emptyset,\{*\}})$ and $\mathbf{CO_{ex}}(\mathbf{Spec}/^+_{\emptyset})$ are equivalent.*

Proof. We follow the same steps as we did in the proofs of Theorem 3.24 and Theorem 3.30. The precise definitions of the functors and natural trasformations that constitute the equivalence are cumbersome, but easy to derive from the transitions we make below.

Exchangeable-output to Entries-only. Given a cyclic operad $\mathcal{O} = (T, \nu^T, \eta_1{}^T, D^T)$ from $\mathbf{CO_{ex}}(\mathbf{Spec}/^+_{\emptyset})$, by Lemma 3.33, we know that $(T, D^T) \simeq (\partial S, \mathtt{ex}_S)$, for some species S from $\mathbf{Spec}/_{\emptyset,\{*\}}$. Together with the definitions of ν^T and η^T, this equivalence gives rise to an operad $(\partial S, \nu^{\partial S}, \eta_1{}^{\partial S}, \mathtt{ex}_S)$, such that $\mathcal{O} \simeq (\partial S, \nu^{\partial S}, \eta_1{}^{\partial S}, \mathtt{ex}_S)$. Since $\int(\partial S, \mathtt{ex}_S) \simeq S$, defining a cyclic operad over the species $\int T$ amounts to defining a cyclic operad $\mathcal{C}_{\mathcal{O}} = (S, \rho_{\nu}{}^S, \eta_2{}^S)$ over the species S. We define $\mathcal{C}_{\mathcal{O}}$ below, whereby we shall write ρ for $\rho_{\nu}{}^S$ and η_2 for $\eta_2{}^S$.

For $X = \emptyset$, we set $\rho_X : (\partial S \cdot \partial S)(X) \to S(X)$ to be the empty function. For $X \neq \emptyset$, observe that defining ρ_X amounts to defining $\rho'_X : \partial(\partial S \cdot \partial S)(X) \to \partial S(X)$. Indeed, in the end, we shall define

$$\rho_X = S(\sigma^{-1}) \circ \rho'_{X\backslash\{x\}} \circ (\partial S \cdot \partial S)(\sigma),$$

where $x \in X$ is arbitrary and $\sigma : X\backslash\{x\} \cup \{*_{X\backslash\{x\}}\} \to X$ renames x to $*_{X\backslash\{x\}}$. For the definition of ρ', we take

$$\rho' = [\rho'_1, \rho'_2] \circ \varphi,$$

where $\rho'_1 : \partial\partial S \cdot \partial S \to \partial S$ and $\rho'_2 : \partial S \cdot \partial\partial S \to \partial S$ are determined by

$$\rho'_1 : \partial\partial S \cdot \partial S \xrightarrow{\mathtt{ex}\cdot id} \partial\partial S \cdot \partial S \xrightarrow{\nu_3} \partial S \qquad \text{and} \qquad \rho'_2 = \rho'_1 \circ \gamma,$$

where ν_3 is as in the axiom (D3), the definition of $[\rho'_1, \rho'_2]$ is set up by Notation 3.5 and φ is the isomorphism from Lemma 3.9(2).

Similarly, defining η_2 amounts to defining $\partial\eta_2 : \partial E_2 \to \partial S$, for which we set

$$\partial\eta_2 = \eta_1{}^{\partial S} \circ \epsilon_2$$

(for the definition of the isomorphism ϵ_2, as well as for the definitions of ex and γ above, see Table 5). We verify the axioms.

(CA1) By Corollary 3.21, the axiom (CA1) for $\mathcal{C}_0$ comes down to the equality $\rho_{21} \circ \theta_1 = \rho_{11}$, whereas the equality $\rho_{21} \circ \theta_1 = \rho_{11}$ clearly follows from the equality $\partial\rho_{21} \circ \partial\theta_1 = \partial\rho_{11}$. We prove the latter equality.

In Diagram 4, the triangle T is the diagram whose commutation we aim to prove and the diagrams L and R are obtained by unfolding the definitions of $\partial\rho_{11}$ and $\partial\rho_{21}$. Finally, the two bottom triangles are obtained by expressing $\partial(\rho' \cdot id)$ in L and R as $\partial([\rho'_1, \rho'_2] \cdot id) \circ \partial(\varphi \cdot id)$.

$$\begin{tikzcd}
 & \partial((\partial\partial S \cdot \partial S) \cdot \partial S) \arrow[rr, "\partial\theta_1"] \arrow[dl, "\partial(i \cdot id)"'] \arrow[dr, "\partial\rho_{11}"'] \arrow[rr, phantom, shift right=5ex, "T"] & & \partial((\partial\partial S \cdot \partial S) \cdot \partial S) \arrow[dl, "\partial\rho_{21}"] \arrow[dr, "\partial(i \cdot id)"] & \\
\partial((\partial\partial S \cdot \partial S + \partial S \cdot \partial\partial S) \cdot \partial S) \arrow[d, "\partial(\varphi^{-1} \cdot id)"'] \arrow[rr, phantom, "L"] & & \partial S \arrow[rr, phantom, "R"] & & \partial((\partial\partial S \cdot \partial S + \partial S \cdot \partial\partial S) \cdot \partial S) \arrow[d, "\partial(\varphi^{-1} \cdot id)"] \\
\partial(\partial(\partial S \cdot \partial S) \cdot \partial S) \arrow[rr, "\partial(\rho' \cdot id)"] \arrow[drr, "\partial(\varphi \cdot id)"'] & & \partial(\partial S \cdot \partial S) \arrow[u, "\rho'"'] & & \partial(\partial(\partial S \cdot \partial S) \cdot \partial S) \arrow[ll, "\partial(\rho' \cdot id)"'] \arrow[dll, "\partial(\varphi \cdot id)"] \\
 & & \partial((\partial\partial S \cdot \partial S + \partial S \cdot \partial\partial S) \cdot \partial S) \arrow[u, "\partial([\rho'_1, \rho'_2] \cdot id)"'] & &
\end{tikzcd}$$

Diagram 4

Snce $\partial(\varphi \cdot id) \circ \partial(\varphi^{-1} \cdot id) = id$ and $\partial([\rho'_1, \rho'_2] \cdot id) \circ \partial(i \cdot id) = \partial(\rho'_1 \cdot id)$, Diagram 4 can be transformed into Diagram 5, in which the triangles L' and R' commute.

$$\begin{tikzcd}
\partial((\partial\partial S \cdot \partial S) \cdot \partial S) \arrow[rr, "\partial\theta_1"] \arrow[dr, "\partial\rho_{11}"] \arrow[ddr, bend right, "\partial(\rho'_1 \cdot id)"'] \arrow[rr, phantom, shift right=5ex, "T"] & & \partial((\partial\partial S \cdot \partial S) \cdot \partial S) \arrow[dl, "\partial\rho_{21}"'] \arrow[ddl, bend left, "\partial(\rho'_1 \cdot id)"] \\
 & \partial S & \\
 & \partial(\partial S \cdot \partial S) \arrow[u, "\rho'"'] \arrow[ul, phantom, "L'"] \arrow[ur, phantom, "R'"] &
\end{tikzcd}$$

Diagram 5.

Therefore, the equality that needs to be proven is

$$A \circ \partial\theta_1 = A, \tag{3.3.10}$$

where $A = \rho' \circ \partial(\rho'_1 \cdot id)$. By implementing $\rho' = [\rho'_1, \rho'_2] \circ \varphi$ in A and then by using the equalities $[\partial\rho'_1 \cdot id, \rho'_1 \cdot \partial id] \circ \varphi = \varphi \circ \partial(\rho'_1 \cdot id)$ and $\partial id = id$, A gets reformulated as $A = [B, C] \circ \varphi$, where

$B=\rho_1'\circ(\partial\rho_1'\cdot id)$ and $C=\rho_2'\circ(\rho_1'\cdot id)$. By setting $\theta=\varphi\circ\partial\theta_1\circ\varphi^{-1}$, the equality (3.3.10) reformulates as

$$[B,C]=[B,C]\circ\Gamma\,. \tag{3.3.11}$$

Since Γ can be expressed by the commutation of Diagram 6.[6], by setting $D=B\circ(\varphi^{-1}\cdot id)\circ(i_l\cdot id)$ and $E=B\circ(\varphi^{-1}\cdot id)\circ(i_r\cdot id)$, the equality 3.3.11 is proven if the following three equalities hold:

$$D=D\circ(\alpha^{-1}\circ(\partial\mathtt{ex}\cdot\gamma)\circ\alpha),\qquad E=C\circ(\alpha^{-1}\circ(\mathtt{ex}\cdot\gamma)\circ\alpha)\qquad\text{and}\qquad C=E\circ(\alpha^{-1}\circ(\mathtt{ex}\cdot\gamma)\circ\alpha).$$

$$\begin{tikzcd}
\partial\partial\partial S\cdot(\partial S\cdot\partial S)+\partial\partial S\cdot(\partial\partial S\cdot\partial S)+\partial\partial S\cdot(\partial S\cdot\partial\partial S) \arrow[r, "\partial\mathtt{ex}\cdot\gamma+\mathtt{ex}\cdot\gamma+\mathtt{ex}\cdot\gamma"] & \partial\partial\partial S\cdot(\partial S\cdot\partial S)+\partial\partial S\cdot(\partial\partial S\cdot\partial S)+\partial\partial S\cdot(\partial S\cdot\partial\partial S) \arrow[d, "\alpha^{-1}+\alpha^{-1}+\alpha^{-1}"] \\
(\partial\partial\partial S\cdot\partial S)\cdot\partial S+(\partial\partial S\cdot\partial\partial S)\cdot\partial S+(\partial\partial S\cdot\partial S)\cdot\partial\partial S \arrow[u, "\alpha+\alpha+\alpha"] & (\partial\partial\partial S\cdot\partial S)\cdot\partial S+(\partial\partial S\cdot\partial\partial S)\cdot\partial S+(\partial\partial S\cdot\partial S)\cdot\partial\partial S \arrow[d, "\varsigma^{-1}+id"] \\
(\partial\partial\partial S\cdot\partial S+\partial\partial S\cdot\partial\partial S)\cdot\partial S+(\partial\partial S\cdot\partial S)\cdot\partial\partial S \arrow[u, "\varsigma+id"] & (\partial\partial\partial S\cdot\partial S+\partial\partial S\cdot\partial\partial S)\cdot\partial S+(\partial\partial S\cdot\partial S)\cdot\partial\partial S \arrow[d, "\varphi^{-1}\cdot id+id"] \\
\partial(\partial\partial S\cdot\partial S)\cdot\partial S+(\partial\partial S\cdot\partial S)\cdot\partial\partial S \arrow[u, "\varphi\cdot id+id"] \arrow[r, "\Gamma"] & \partial(\partial\partial S\cdot\partial S)\cdot\partial S+(\partial\partial S\cdot\partial S)\cdot\partial\partial S
\end{tikzcd}$$

Diagram 6

Therefore, the first equality that needs to be proven is

$$\rho_1'\circ(\partial\rho_1'\cdot id)\circ(\varphi^{-1}\cdot id)\circ(i_l\cdot id)=\rho_1'\circ(\partial\rho_1'\cdot id)\circ(\varphi^{-1}\cdot id)\circ(i_l\cdot id)\circ(\alpha^{-1}\circ(\partial\mathtt{ex}\cdot\gamma)\circ\alpha),$$

and the outer part of Diagram 7 corresponds exactly to this equality once the definition of ρ' (via ν) is unfolded.

$$\begin{tikzcd}
(\partial\partial\partial S\cdot\partial S)\cdot\partial S \arrow[r, "\alpha"] \arrow[d, "(\varphi^{-1}\circ i_l)\cdot id"'] \arrow[ddr, "(\partial\mathtt{ex}\cdot id)\cdot id"] & \partial\partial\partial S\cdot(\partial S\cdot\partial S) \arrow[rr, "\partial\mathtt{ex}\cdot\gamma"] & & \partial\partial\partial S\cdot(\partial S\cdot\partial S) \arrow[r, "\alpha^{-1}"] & (\partial\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "(\varphi^{-1}\circ i_l)\cdot id"] \arrow[ddl, "(\partial\mathtt{ex}\cdot id)\cdot id"'] \\
\partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial(\mathtt{ex}\cdot id)\cdot id"'] \arrow[r, phantom, "I_l"] & & M & & \partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial(\mathtt{ex}\cdot id)\cdot id"] \arrow[l, phantom, "I_r"] \\
\partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial\nu\cdot id"'] & (\partial\partial\partial S\cdot\partial S)\cdot\partial S \arrow[l, "(\varphi^{-1}\circ i_l)\cdot id"'] \arrow[rr, "\alpha^{\cdot -1}\circ(\partial\mathtt{ex}\cdot c)\circ\alpha^{\cdot}"] \arrow[d, "(\partial\mathtt{ex}\circ\mathtt{ex}\circ\partial\mathtt{ex})\cdot id"'] & & (\partial\partial\partial S\cdot\partial S)\cdot\partial S \arrow[r, "(\varphi^{-1}\circ i_l)\cdot id"] \arrow[d, "(\mathtt{ex}\circ\partial\mathtt{ex}\circ\mathtt{ex})\cdot id"] & \partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial\nu\cdot id"] \\
\partial\partial S\cdot\partial S \arrow[d, "\mathtt{ex}\cdot id"'] \arrow[r, phantom, "J_l"] & (\partial\partial\partial S\cdot\partial S)\cdot\partial S \arrow[rr, "\alpha^{\cdot -1}\circ(\mathtt{ex}\cdot c)\circ\alpha^{\cdot}"] \arrow[d, "(\varphi^{-1}\circ i_l)\cdot id"'] \arrow[rr, phantom, "L", shift left=20] & & (\partial\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "(\varphi^{-1}\circ i_l)\cdot id"] & \partial\partial S\cdot\partial S \arrow[d, "\mathtt{ex}\cdot id"] \arrow[l, phantom, "J_r"] \\
\partial\partial S\cdot\partial S \arrow[ddrr, "\nu"'] & \partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[l, "\partial\nu\cdot id"'] & K & \partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[r, "\partial\nu\cdot id"] & \partial\partial S\cdot\partial S \arrow[ddll, "\nu"] \\
& & & & \\
& & \partial S & &
\end{tikzcd}$$

Diagram 7.

The rest of the arrows show that the outer part indeed commutes. Notice that

[6] In the top horizontal arrow of Diagram 6, the first $\mathtt{ex}\cdot\gamma$ maps the second summand on the left to the third one on the right, and the second $\mathtt{ex}\cdot\gamma$ maps the third summand on the left to the second one on the right.

- J_l and J_r commute, since they become the commuting squares of Remark 3.32 and (D3), respectively, once the two sequences of morphisms defining ν_3 are "wrapped up" and D is set to be ex,
- K commutes, as it represents the equality $\nu_{21} \circ \beta_1 = \nu_{11}$ of the axiom [OA1], and
- I_l, I_r, L and M commute as instances of the naturality conditions for φ and α.

The second equality is

$$\rho_1' \circ (\partial \rho_1' \cdot id) \circ (\varphi^{-1} \cdot id) \circ (i_r \cdot id) = \rho_2' \circ (\rho_1' \cdot id) \circ \alpha^{-1} \circ (\mathtt{ex} \cdot \gamma) \circ \alpha,$$

and the corresponding diagram is (the outer part of) Diagram 8. This diagram commutes because

- I becomes the commuting pentagon of (D4), once the two sequences of morphisms definition of ν_4 are "wrapped up" and D is set to be ex, and
- J commutes, as it corresponds to the equality $\nu_{22} \circ \beta_2 = \nu_{12}$ of the axiom [OA1].

$$\begin{tikzcd}
(\partial\partial S \cdot \partial\partial S) \cdot \partial S \arrow[r, "\alpha^{\cdot}"] \arrow[d, "(\varphi^{-1} \circ i_r) \cdot id"'] \arrow[dr, "\alpha"] \arrow[ddr, "\gamma \cdot id"] \arrow[dddr, "(\mathtt{ex} \cdot id) \cdot id"] & \partial\partial S \cdot (\partial\partial S \cdot \partial S) \arrow[rr, "\mathtt{ex} \cdot \gamma"] & & \partial\partial S \cdot (\partial S \cdot \partial\partial S) \arrow[r, "\alpha^{\cdot -1}"] & (\partial\partial S \cdot \partial S) \cdot \partial\partial S \arrow[d, "(\mathtt{ex} \cdot id) \cdot id"] \\
\partial(\partial\partial S \cdot \partial S) \cdot \partial S \arrow[dd, "\partial(\mathtt{ex} \cdot id) \cdot id"'] & \partial\partial S \cdot (\partial\partial S \cdot \partial S) \arrow[rr, "id \cdot \gamma"] & & \partial\partial S \cdot (\partial S \cdot \partial\partial S) \arrow[r, "\alpha^{\cdot -1}"] & (\partial\partial S \cdot \partial S) \cdot \partial\partial S \arrow[dd, "\nu \cdot id"] \arrow[ddl, "\gamma"'] \\
 & (\partial\partial S \cdot \partial\partial S) \cdot \partial S \arrow[drr, "\alpha"] & & & \\
\partial(\partial\partial S \cdot \partial S) \cdot \partial S \arrow[dd, "\partial\nu \cdot id"'] & (\partial\partial S \cdot \partial\partial S) \cdot \partial S \arrow[l, "(\varphi^{-1} \circ i_r) \cdot id"'] \arrow[d, "(\mathtt{ex} \cdot \mathtt{ex}) \cdot id"'] & & \partial\partial S \cdot (\partial\partial S \cdot \partial S) \arrow[ddr, "id \cdot \nu"] \arrow[ddd, "\mathtt{ex} \cdot id"] & \partial S \cdot \partial\partial S \arrow[dd, "\gamma"] \\
 & (\partial\partial S \cdot \partial\partial S) \cdot \partial S \arrow[dd, "\gamma \cdot id"'] & & & \\
\partial\partial S \cdot \partial S \arrow[dd, "\mathtt{ex} \cdot id"'] \arrow[r, phantom, "I"] & & & & \partial\partial S \cdot \partial S \arrow[dd, "\mathtt{ex} \cdot id"] \\
 & (\partial\partial S \cdot \partial\partial S) \cdot \partial S \arrow[rr, "\alpha"] \arrow[d, "(\varphi^{-1} \circ i_r) \cdot id"'] & & \partial\partial S \cdot (\partial\partial S \cdot \partial S) \arrow[dr, "id \cdot \nu"] & \\
\partial\partial S \cdot \partial S \arrow[drr, "\nu"'] & \partial(\partial\partial S \cdot \partial S) \cdot \partial S \arrow[l, "\partial\nu \cdot id"'] \arrow[rr, phantom, "J"] & & & \partial\partial S \cdot \partial S \arrow[dll, "\nu"] \\
 & & \partial S & &
\end{tikzcd}$$

Diagram 8

The last equality is

$$\rho_2' \circ (\rho_1' \cdot id) = \rho_1' \circ (\partial \rho_1' \cdot id) \circ (\varphi^{-1} \cdot id) \circ (i_r \cdot id) \circ \alpha^{-1} \circ (\mathtt{ex} \cdot \gamma) \circ \alpha,$$

and it follows from the second one, since $(\mathtt{ex} \cdot \gamma)^{-1} = \mathtt{ex} \cdot \gamma$.

(CA2) By an analysis similar to the one we made for (CA1), it can be shown that (CA2) follows from the equalities

$$\rho_1' \circ (\partial\partial\eta_{\mathcal{C}} \cdot id) = \partial\pi_1 \circ \partial\lambda^{\blacktriangle} \circ (\varphi^{-1} \circ i_l) : \partial\partial E_2 \cdot \partial S \to \partial S \tag{3.3.12}$$

and

$$\rho_2' \circ (\partial\eta_{\mathcal{C}} \cdot id) = \partial\pi_1 \circ \partial\lambda^{\blacktriangle} \circ (\varphi^{-1} \circ i_r) : \partial E_2 \cdot \partial\partial S \to \partial S. \tag{3.3.13}$$

In Diagram 9, the inner triangle represents the equation (3.3.12). It commutes by the commutations of the three diagrams that surround it (easy to check) and from the commutation of the outer triangle, which represents the left triangle from the axiom [OA2]. The equality (3.3.13) is verified by a similar diagram, whose outer part will commute as the right triangle from [OA2].

$$\begin{tikzcd}
\partial E_1\cdot\partial S \arrow[rrrr, "\eta_{\mathcal{O}}\cdot id"] \arrow[ddddrr, "\lambda^{\star}"'] & & & & \partial\partial S\cdot\partial S \arrow[ddddll, "\nu"] \\
& \partial\partial E_2\cdot\partial S \arrow[ul, "\partial\epsilon_2\cdot id"'] \arrow[rr, "\partial\partial\eta_{\mathcal{C}}\cdot id"] \arrow[d, "\varphi^{-1}\circ i_l"] & & \partial\partial S\cdot\partial S \arrow[ur, "\mathtt{ex}\cdot id"] \arrow[dddl, "\rho_1"'] & \\
& \partial(\partial E_2\cdot\partial S) \arrow[dr, "\partial\lambda^{\blacktriangle}"] & D_1 & & \\
& & \partial S^{\bullet} \arrow[d, "\partial\pi_1"] & & \\
& & \partial S & &
\end{tikzcd}$$

Diagram 9

Entries-only to Exchangeable-output. Given an entries-only cyclic operad $\mathcal{C}=(S,\rho^S,\eta_2{}^S)$, we the corresponing exchangeable-output cyclic operad $\mathcal{O}_{\mathcal{C}}=(\partial S,\nu_{\rho}{}^{\partial S},\eta_1{}^{\partial S},\mathtt{ex}_S)$ is defined by introducing $\nu_{\rho}{}^{\partial S}:\partial S\star\partial S\to\partial S$ (ν for short) as

$$\nu:\partial\partial S\cdot\partial S\xrightarrow{\mathtt{ex}\cdot id}\partial\partial S\cdot\partial S\xrightarrow{i_l}\partial\partial S\cdot\partial S+\partial S\cdot\partial\partial S\xrightarrow{\varphi^{-1}}\partial(\partial S\cdot\partial S)\xrightarrow{\partial\rho}\partial S,$$

and $\eta_1{}^{\partial S}:E_1\to\partial S$ (η_1 for short) as $\eta_1=\partial\eta_2{}^S\circ\epsilon_2^{-1}$. Recall that the isomorphism $\mathtt{ex}_S$ arises a priori by Lemma 3.33, and therefore, it automatically satisfies the axioms (D1) and (D2). The axiom (D3) is then verified easily by using (D2), naturality of φ and naturality of $\mathtt{ex_S}$, and the axiom (D4) follows essentially by Lemma 3.20. We now indicate how to verify the remaining axioms.

[OA1] The outer part of Diagram 10 represents the equality $\nu_{21}\circ\beta_1=\nu_{11}$ (once the definition of ν via ρ is unfolded). The proof that it commutes uses

- the commutation of the diagram E, where

$$\psi=(\varphi^{-1}\circ i_l)\circ((\varphi^{-1}\circ i_l)\cdot id)\circ((\mathtt{ex}\cdot id)\cdot id)\circ((\partial\mathtt{ex}\cdot id)\cdot id),$$

  which follows by the equality $\partial\mathtt{ex}\circ\mathtt{ex}\circ\partial\mathtt{ex}=\mathtt{ex}\circ\partial\mathtt{ex}\circ\mathtt{ex}$,
- the commutation of the diagram G, representing the equality $\partial\rho_{21}\circ\partial\theta_1=\partial\rho_{11}$, which, in turn, holds by (CA1),
- the commutations of F_1 and F_2, which follow by the naturality of φ, and
- the commutations of R_1 and R_2, which follow by the naturality of ρ.

The outer part of Diagram 11, which represents the equality $\nu_{22}\circ\beta_2=\nu_{12}$, commutes by

- the commutation of the diagram E, where

$$\phi_1=\varphi^{-1}\circ i_l\circ(\varphi^{-1}\circ i_l\circ(\mathtt{ex}\cdot id))\cdot id\quad\text{ and }\quad\phi_2=\varphi^{-1}\circ i_l\circ(\mathtt{ex}\cdot id),$$

  which follows by the naturality of θ_2 (notice that $\theta_2=(id\cdot(\mathtt{ex}\cdot id))\circ\alpha$),
- the commutation of G, representing the equality $\partial\rho_{22}\circ\partial\theta_2=\partial\rho_{12}$, which, in turn, holds by (CA1), and
- the commutations of F_1, F_2, R_1 and R_2, which are of the same kind as in Diagram 10.

Notice that the equality $\nu_{23}\circ\beta_3=\nu_{13}$ also follows by the commutation of Diagram 11.

$$\begin{tikzcd}[column sep=small]
(\partial\partial\partial S\cdot\partial S)\cdot\partial S \arrow[rr, "\alpha^{\cdot}"] \arrow[d, "(\varphi^{-1}\circ i_l)\cdot id"'] \arrow[rrddd, "\psi"] & & \partial\partial\partial S\cdot(\partial S\cdot\partial S) \arrow[rr, "\mathtt{ex}\cdot\gamma"] & & \partial\partial\partial S\cdot(\partial S\cdot\partial S) \arrow[rr, "\alpha^{\cdot -1}"] & & (\partial\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "(\varphi^{-1}\circ i_l)\cdot id"] \arrow[llddd, "\psi"'] \\
\partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial(\mathtt{ex}\cdot id)\cdot id"'] & & & E & & & \partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial(\mathtt{ex}\cdot id)\cdot id"] \\
\partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial(\varphi^{-1}\circ i_l)\cdot id"'] & & & & & & \partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial(\varphi^{-1}\circ i_l)\cdot id"] \\
\partial\partial(\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\mathtt{ex}\cdot id"'] & F_1 & \partial((\partial\partial S\cdot\partial S)\cdot\partial S) \arrow[rr, "\partial\theta_1"] \arrow[dd, "\partial((\varphi^{-1}\circ i_l)\cdot id)"] & & \partial((\partial\partial S\cdot\partial S)\cdot\partial S) \arrow[dd, "\partial((\varphi^{-1}\circ i_l)\cdot id)"'] & F_2 & \partial\partial(\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\mathtt{ex}\cdot id"] \\
\partial\partial(\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial\partial\rho\cdot id"'] \arrow[rrd, "\varphi^{-1}\circ i_l"] & & & & & & \partial\partial(\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial\partial\rho\cdot id"] \arrow[lld, "\varphi^{-1}\circ i_l"'] \\
\partial\partial S\cdot\partial S \arrow[d, "\varphi^{-1}\circ i_l"'] & R_1 & \partial(\partial(\partial S\cdot\partial S)\cdot\partial S) \arrow[lld, "\partial(\partial\rho\cdot id)"] & G & \partial(\partial(\partial S\cdot\partial S)\cdot\partial S) \arrow[rrd, "\partial(\partial\rho\cdot id)"'] & R_2 & \partial\partial S\cdot\partial S \arrow[d, "\varphi^{-1}\circ i_l"] \\
\partial(\partial S\cdot\partial S) \arrow[rrr, "\partial\rho"'] & & & \partial S & & & \partial(\partial S\cdot\partial S) \arrow[lll, "\partial\rho"]
\end{tikzcd}$$

Diagram 10

$$\begin{tikzcd}[column sep=small]
(\partial\partial S\cdot\partial\partial S)\cdot\partial S \arrow[rrrrrr, "\alpha"] \arrow[d, "(\varphi^{-1}\circ i_r)\cdot id"'] \arrow[rrddd, "\phi_1"] & & & & & & \partial\partial S\cdot(\partial\partial S\cdot\partial S) \arrow[d, "id\cdot(\mathtt{ex}\cdot id)"] \\
\partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial(\mathtt{ex}\cdot id)\cdot id"'] & & & E & & & \partial\partial S\cdot(\partial\partial S\cdot\partial S) \arrow[d, "id\cdot(\varphi^{-1}\circ i_l)"] \arrow[lldd, "\phi_2"'] \\
\partial(\partial\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial(\varphi^{-1}\circ i_l)\cdot id"'] & & & & & & \partial\partial S\cdot\partial(\partial S\cdot\partial S) \arrow[d, "\mathtt{ex}\cdot id"] \\
\partial\partial(\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\mathtt{ex}\cdot id"'] & F_1 & \partial((\partial S\cdot\partial\partial S)\cdot\partial S) \arrow[rr, "\partial\theta_2"] \arrow[dd, "\partial((\varphi^{-1}\circ i_r)\cdot id)"] & & \partial(\partial S\cdot(\partial\partial S\cdot\partial S)) \arrow[dd, "\partial(id\cdot(\varphi^{-1}\circ i_l))"'] & F_2 & \partial\partial S\cdot\partial(\partial S\cdot\partial S) \arrow[d, "id\cdot\partial\rho"] \arrow[lldd, "\varphi^{-1}\circ i_l"'] \\
\partial\partial(\partial S\cdot\partial S)\cdot\partial S \arrow[d, "\partial\partial\rho\cdot id"'] \arrow[rrd, "\varphi^{-1}\circ i_l"] & & & & & R_2 & \partial\partial S\cdot\partial S \arrow[d, "\varphi^{-1}\circ i_l"] \\
\partial\partial S\cdot\partial S \arrow[d, "\varphi^{-1}\circ i_l"'] & R_1 & \partial(\partial(\partial S\cdot\partial S)\cdot\partial S) \arrow[lld, "\partial(\partial\rho\cdot id)"] & G & \partial(\partial S\cdot\partial(\partial S\cdot\partial S)) \arrow[rr, "\partial(id\cdot\partial\rho)"] & & \partial(\partial S\cdot\partial S) \arrow[llld, "\partial\rho"] \\
\partial(\partial S\cdot\partial S) \arrow[rrr, "\partial\rho"'] & & & \partial S & & &
\end{tikzcd}$$

Diagram 11

[OA2] The commutation of the left triangle from [OA2] follows by replacing ν in Diagram 9 with its definition via ρ and by setting $\rho_1 = \partial\rho\circ\varphi^{-1}\circ i_l$. The commutation of the right triangle follows analogously, relative to the diagram that arises in the proof of the equality (3.3.13).

The isomorphism of cyclic operads $\mathcal{C}$ and $\mathcal{C}_{\mathcal{O}_{\mathcal{C}}}$ (and $\mathcal{O}$ and $\mathcal{O}_{\mathcal{C}_{\mathcal{O}}}$). As it was the case in the proof of Theorem 3.30, the isomorphism at the level of the underlying species exists by Lemma 3.33. The first isomorphism of cyclic operads follows from the equalities[7]

$$\partial\eta_2{}^S = \partial\eta_2{}^S\circ\epsilon_2^{-1}\circ\epsilon_2 \quad\text{ and }\quad \partial\rho^S = \rho'_{\nu_\rho^{\partial S}}.$$

[7] These would be isomorphisms (rather than equalities) if we considered the sequence $S\to\partial S\to\int\partial S$ instead of $S\to\partial S\to S$.

The first equality is obvoiusly trivial, while, since $\rho'_{\nu_\rho^{\partial S}} \circ \varphi^{-1} = [\rho'_{1\nu_\rho^{\partial S}}, \rho'_{2\nu_\rho^{\partial S}}]$, the second equality follows from the equalities

$$\partial\rho^S \circ \varphi^{-1} \circ i_l = \rho'_{1\nu_\rho^{\partial S}} \quad \text{and} \quad \partial\rho^S \circ \varphi^{-1} \circ i_r = \rho'_{2\nu_\rho^{\partial S}}.$$

These latter two equalities are verified as follows:

$$\begin{aligned} \rho'_{1\nu_\rho^{\partial S}} &= \nu_\rho^{\partial S} \circ (\mathtt{ex} \cdot id_{\partial S}) \\ &= \partial\rho^S \circ \varphi^{-1} \circ i_l \circ (\mathtt{ex} \cdot id_{\partial S}) \circ (\mathtt{ex} \cdot id_{\partial S}) \\ &= \partial\rho^S \circ \varphi^{-1} \circ i_l \end{aligned}$$

and

$$\begin{aligned} \rho'_{2\nu_\rho^{\partial S}} &= \nu_\rho^{\partial S} \circ (\mathtt{ex} \cdot id_{\partial S}) \circ \gamma \\ &= \partial\rho^S \circ \varphi^{-1} \circ i_l \circ (\mathtt{ex} \cdot id_{\partial S}) \circ (\mathtt{ex} \cdot id_{\partial S}) \circ \mathtt{c} \\ &= \partial\rho^S \circ \varphi^{-1} \circ i_l \circ \mathtt{c} \\ &= \partial\rho^S \circ \partial\gamma \circ \varphi^{-1} \circ i_r \\ &= \partial\rho^S \circ \varphi^{-1} \circ i_r\,. \end{aligned}$$

The second isomorphism follows from the equalities[8]

$$\partial\eta_1^{\partial S} = \partial(\eta_1^{\partial S} \circ \epsilon_2) \circ \epsilon_2^{-1} \quad \text{and} \quad \nu = \nu_{\rho_\nu}.$$

The equality $\nu = \nu_{\rho_\nu}$ is verified as follows:

$$\begin{aligned} \nu_{\rho_\nu} &= \partial\rho_\nu \circ \varphi^{-1} \circ i_l \circ (\mathtt{ex} \cdot id) \\ &= [\nu \circ (\mathtt{ex} \cdot id_{\partial S}), \nu \circ (\mathtt{ex} \cdot id_{\partial S}) \circ \gamma] \circ \varphi \circ \varphi^{-1} \circ i_l \circ (\mathtt{ex} \cdot id) \\ &= \nu \circ (\mathtt{ex} \cdot id_{\partial S}) \circ (\mathtt{ex} \cdot id) \\ &= \nu. \end{aligned}$$

This completes the proof. ∎

[8]Like before, these would be isomorphisms if we considered the sequence $S \to \int S \to \partial \int S$.

Chapter 4

Categorified cyclic operads

In the last chapter of this thesis, we make a leap from cyclic operads defined in set-theoretic terms, which we considered up to now, to their categorifications. The process of categorification that we carry out is outlined by the replacement of set-theoretic concepts from Definition 1.4 (entries-only) and Definition 3.25 (exchangeable-output) of cyclic operads with their category-theoretic analogues, as indicated in the Introduction.

For the sake of simplicity, we shall not consider units in neither of these definitions. For both of them, the removal of units is made simply by forgetting their structure and omitting the unitality axioms (see Remark 1.3 and Remark 1.9). Therefore, in the remaining of this chapter, we shall work with the following non-unital axiomatisations of cyclic operads:

$$\texttt{(A1)}, \texttt{(CO)} \text{ and } \texttt{(EQ)},$$

for entries-only cyclic operads, and

$$\texttt{[A1]}, \texttt{[A2]}, \texttt{[EQ]}, \texttt{[DIN]}, \texttt{[DEQ]}, \texttt{[DEX]}, \texttt{[DC1]} \text{ and } \texttt{[DC2]},$$

for exchangeable-output cyclic operads.

This chapter is organised as follows. In Section 4.1, we introduce categorified entries-only cyclic operads by relaxing the axioms `(A1)` and `(CO)` to isomorphisms, while leaving equivariance strict, and by formulating conditions which ensure the coherence of the obtained isomorphisms. We examine the "operadic" properties of the obtained categorification, essential for reducing the coherence problem to the coherence of weak Cat-operads of [DP15]. The largest part of the section will be devoted to the proof of the coherence theorem, whose statement "all canonical diagrams commute" we make precise by introducing the syntax of canonical diagrams. As we indicated in the Introduction, the proof is obtained by restricting the coherence problem from the class of all canonical diagrams to the class of canonical diagrams of [DP15], by three consecutive faithful reductions: we first "remove" symmetries, then "cyclicity" and, finally, we pass from the non-skeletal to the skeletal framework. Section 4.2 deals with (non-skeletal) exchangeable-output categorified cyclic operads. The choice of the axioms which should be weakened in this case is predetermined by the proof of Theorem 3.30: if we want to preserve the equivalence of entries-only and exchangeable-output cyclic operads in the categorified setting, then these axioms must be `[A1]`, `[A2]` and `[DC2]`. By using the proof of Theorem 3.30 as a "dictionary" from entries-only to exchangeable-output cyclic operads, the coherence conditions for the three obtained isomorphisms are established by translating to the exchangeable-output language the coherences of the entries-only categorification, which is not trivial. Finally, by adapting the proof of Theorem 3.30 for the non-unital and categorified setting, we give a proof of the equivalence between the exchangeable-output and the entries-only categorified cyclic operads, which guarantees the coherence of the former notion. We round up the chapter with a comment on skeletal exchangeable-output categorified cyclic operads.

In the remainder of the chapter, we shall use latin letters for operations of a categorified cyclic operad, and greek letters for morphisms between them.

4.1 Categorified entries-only cyclic operads

This section deals with categorified entries-only cyclic operads. In the first part of the section, we introduce the categorified notion and exhibit important properties. The second one is dedicated to the proof of the coherence theorem.

4.1.1 The definition and properties

Chasing coherence of sequential associativity `(A1)` and commutativity `(CO)`, relaxed to isomorphisms, led us to the following definition.

Definition 4.1. A *categorified entries-only cyclic operad* is a functor $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Cat}$, together with

- a family of bifunctors

$$ {}_{x}\circ_{y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash\{x\} \cup Y\backslash\{y\}), $$

  called *partial composition operations of* $\mathcal{C}$, indexed by arbitrary non-empty finite sets X and Y and elements $x \in X$ and $y \in Y$, such that $X\backslash\{x\} \cap Y\backslash\{y\} = \emptyset$, which are subject to the equivariance axiom `(EQ)`, and

- two natural isomorphisms, β and γ, called the *associator* and the *commutator*, whose respective components

$$ \beta^{x,\underline{x};y,\underline{y}}_{f,g,h} : (f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h \to f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{y}\circ_{\underline{y}}\,h) \quad \text{and} \quad \gamma^{x,y}_{f,g} : f\,{}_{x}\circ_{y}\,g \to g\,{}_{y}\circ_{x}\,f\,, $$

  are natural in f, g and h, and are subject to the following coherence conditions:

  - (β-pentagon)

$$ \begin{tikzcd} & ((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k \arrow[dl, "\beta^{x,\underline{x};y,\underline{y}}_{f,g,h}\,{}_{z}\circ_{\underline{z}}\,1_k"'] \arrow[dr, "\beta^{y,\underline{y};z,\underline{z}}_{f\,{}_{x}\circ_{\underline{x}}g,h,k}"] & \\ (f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{y}\circ_{\underline{y}}\,h))\,{}_{z}\circ_{\underline{z}}\,k \arrow[d, "\beta^{x,\underline{x};z,\underline{z}}_{f,g\,{}_{y}\circ_{\underline{y}}h,k}"'] & & (f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,(h\,{}_{z}\circ_{\underline{z}}\,k) \arrow[d, "\beta^{x,\underline{x};y,\underline{y}}_{f,g,h\,{}_{z}\circ_{\underline{z}}k}"] \\ f\,{}_{x}\circ_{\underline{x}}\,((g\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k) \arrow[rr, "1_f\,{}_{x}\circ_{\underline{x}}\,\beta^{y,\underline{y};z,\underline{z}}_{g,h,k}"] & & f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{y}\circ_{\underline{y}}\,(h\,{}_{z}\circ_{\underline{z}}\,k)) \end{tikzcd} $$

  - ($\beta\gamma$-hexagon)

$$ \begin{tikzcd} (f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h \arrow[r, "\beta^{x,\underline{x};y,\underline{y}}_{f,g,h}"] \arrow[d, "\gamma^{x,\underline{x}}_{f,g}\,{}_{y}\circ_{\underline{y}}\,1_h"'] & f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{y}\circ_{\underline{y}}\,h) \arrow[r, "\gamma^{x,\underline{x}}_{f,g\,{}_{y}\circ_{\underline{y}}h}"] & (g\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{\underline{x}}\circ_{x}\,f \arrow[d, "\gamma^{y,\underline{y}}_{g,h}\,{}_{\underline{x}}\circ_{x}\,1_f"] \\ (g\,{}_{\underline{x}}\circ_{x}\,f)\,{}_{y}\circ_{\underline{y}}\,h \arrow[r, "\gamma^{y,\underline{y}}_{g\,{}_{\underline{x}}\circ_{x}f,h}"] & h\,{}_{\underline{y}}\circ_{y}\,(g\,{}_{\underline{x}}\circ_{x}\,f) & (h\,{}_{\underline{y}}\circ_{y}\,g)\,{}_{\underline{x}}\circ_{x}\,f \arrow[l, "\beta^{\underline{y},y;\underline{x},x}_{h,g,f}"'] \end{tikzcd} $$

- $(\beta\gamma$`-decagon)`

$$\begin{tikzcd}[column sep=small]
& (h\,{}_{\underline{y}}\circ_{y}(f\,{}_{x}\circ_{\underline{x}}\,g))\,{}_{z}\circ_{\underline{z}}\,k \arrow[r, "\beta^{\underline{y},y;z,\underline{z}}_{h,f\,{}_{x}\circ_{\underline{x}}g,k}"] & h\,{}_{\underline{y}}\circ_{y}((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k) \arrow[dr, "\gamma^{\underline{y},y}_{h,(f\,{}_{x}\circ_{\underline{x}}g)\,{}_{z}\circ_{\underline{z}}k}"] & \\
((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k \arrow[ur, "\gamma^{y,\underline{y}}_{f\,{}_{x}\circ_{\underline{x}}g,h}\,{}_{z}\circ_{\underline{z}}\,1_k"] \arrow[d, "\beta^{x,\underline{x};y,\underline{y}}_{f,g,h}\,{}_{z}\circ_{\underline{z}}\,1_k"'] & & & ((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h \arrow[d, "\beta^{x,\underline{x};z,\underline{z}}_{f,g,k}\,{}_{y}\circ_{\underline{y}}\,1_h"] \\
(f\,{}_{x}\circ_{\underline{x}}(g\,{}_{y}\circ_{\underline{y}}\,h))\,{}_{z}\circ_{\underline{z}}\,k \arrow[d, "\beta^{x,\underline{x};z,\underline{z}}_{f,g\,{}_{y}\circ_{\underline{y}}h,k}"'] & & & (f\,{}_{x}\circ_{\underline{x}}(g\,{}_{z}\circ_{\underline{z}}\,k))\,{}_{y}\circ_{\underline{y}}\,h \arrow[d, "\beta^{x,\underline{x};y,\underline{y}}_{f,g\,{}_{z}\circ_{\underline{z}}k,h}"] \\
f\,{}_{x}\circ_{\underline{x}}((g\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k) \arrow[dr, "1_{f\,{}_{x}\circ_{\underline{x}}}(\gamma^{y,\underline{y}}_{g,h}\,{}_{z}\circ_{\underline{z}}\,1_k)"'] & & & f\,{}_{x}\circ_{\underline{x}}((g\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h) \\
& f\,{}_{x}\circ_{\underline{x}}((h\,{}_{\underline{y}}\circ_{y}\,g)\,{}_{z}\circ_{\underline{z}}\,k) \arrow[r, "1_{f\,{}_{x}\circ_{\underline{x}}}\beta^{\underline{y},y;z,\underline{z}}_{h,g,k}"] & f\,{}_{x}\circ_{\underline{x}}(h\,{}_{\underline{y}}\circ_{y}(g\,{}_{z}\circ_{\underline{z}}\,k)) \arrow[ur, "1_{f\,{}_{x}\circ_{\underline{x}}}\gamma^{\underline{y},y}_{h,g\,{}_{z}\circ_{\underline{z}}k}"'] &
\end{tikzcd}$$

- $(\gamma$`-involution)`

$$\begin{tikzcd}
f\,{}_{x}\circ_{\underline{x}}\,g \arrow[d, "\gamma^{x,\underline{x}}_{f,g}"'] \arrow[dr, "1_{f\,{}_{x}\circ_{\underline{x}}g}"] & \\
f\,{}_{\underline{x}}\circ_{x}\,g \arrow[r, "\gamma^{\underline{x},x}_{g,f}"] & f\,{}_{x}\circ_{\underline{x}}\,g
\end{tikzcd}$$

where $1_{(-)}$ denotes the identity morphism for $(-)$, as well as the following conditions which involve the action of $\mathcal{C}(\sigma)$, where $\sigma : Y \to X$, on the morphisms of $\mathcal{C}(X)$:

- $(\beta\sigma)$ if the equality $((f\,{}_{x}\circ_{\underline{x}}\;g)\,{}_{y}\circ_{\underline{y}}\,h)^{\sigma} = (f^{\sigma_1}\,{}_{x'}\circ_{\underline{x}'}\;g^{\sigma_2})\,{}_{y'}\circ_{\underline{y}'}\,h^{\sigma_3}$ holds by `(EQ)`, then

$$(\beta^{x,\underline{x};y,\underline{y}}_{f,g,k})^{\sigma} = \beta^{x',\underline{x}';y',\underline{y}'}_{f^{\sigma_1},g^{\sigma_2},h^{\sigma_3}},$$

- $(\gamma\sigma)$ if the equality $(f\,{}_{x}\circ_{y}\;g)^{\sigma} = f^{\sigma_1}\,{}_{x'}\circ_{y'}\;g^{\sigma_2}$ holds by `(EQ)`, then

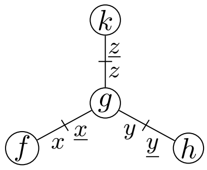
$$(\gamma^{x,y}_{f,g})^{\sigma} = \gamma^{x',y'}_{f^{\sigma_1},g^{\sigma_2}},$$

- `(EQ-mor)` if the equality $(f\,{}_{x}\circ_{y}\;g)^{\sigma} = f^{\sigma_1}\,{}_{x'}\circ_{y'}\;g^{\sigma_2}$ holds by `(EQ)`, and if $\varphi : f \to f'$ and $\psi : g \to g'$, then

$$(\varphi\,{}_{x}\circ_{y}\,\psi)^{\sigma} = \varphi^{\sigma_1}\,{}_{x'}\circ_{y'}\,\psi^{\sigma_2}.$$

□

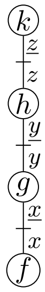
Remark 4.2. *The nodes of the diagrams of Definition 4.1 can be viewed as formal expressions built over operations $f, g, \ldots$ and their entries $x, \underline{x}, y, \underline{y}, \ldots$. For each diagram, the rules for assembling correctly these expressions are determined by the "origin of entries", i.e. by the uniquely determined relation between the involved operations and entries, whose instances have the form "x is an entry of f". For example, in $(\beta\gamma$`-decagon)`, the legitimacy of all the nodes in the diagram witnesses that x is entry of f, $\underline{x}$, y and z are entries of g, $\underline{y}$ is the entry of h and $\underline{z}$ is the entry of k. From the tree-wise perspective, these data can be encoded by the unrooted tree*

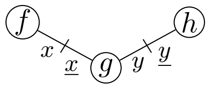


This tree also illustrates the fact that the morphism, say,

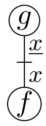
$$\beta^{\underline{x},x;y,\underline{y}}_{g,f,h} : (g\,{}_{\underline{x}}\circ_x f)\,{}_y\circ_{\underline{y}} h \to g\,{}_{\underline{x}}\circ_x (f\,{}_y\circ_{\underline{y}} h)$$

does not exist (for these particular f, g and h), since its codomain is not well-formed, which exemplifies the difference between the setting of symmetric monoidal categories, where an instance of the associator exists for any (ordered) triple of objects*. The trees corresponding to* `(`β`-pentagon)`, `(`$\beta\gamma$`-hexagon)` *and* `(`γ`-involution)` *are*





and



respectively. In §4.1.4, we shall introduce a formal tree-wise representation of the operations of a categorified cyclic operads, based on this intuition. Until then, we shall continue to omit the data about the "origin of entries" whenever possible.

Remark 4.3. *Observe that, for a categorified cyclic operad $\mathcal{C}$ and a finite set X, both the objects and the morphisms of $\mathcal{C}(X)$ enjoy equivariance: at the level of objects, this is ensured by* `(EQ)`*, and at the level of morphisms, by* `(EQ-mor)`*.*

In the remainder of the section, we shall work with a fixed categorified entries-only cyclic operad $\mathcal{C}$. In the remark that follows, we list the equalities on objects and morphisms of $\mathcal{C}(X)$ which are implicitly imposed by the structure of $\mathcal{C}$.

Remark 4.4. *For an arbitrary finite set X, the following equalities hold in $\mathcal{C}(X)$:*

1. *the categorical equations:*

   a) $\varphi\circ 1_f = \varphi = 1_g\circ\varphi$, *for* $\varphi : f\to g$,

   b) $(\varphi\circ\phi)\circ\psi = \varphi\circ(\phi\circ\psi)$,

2. *the equations imposed by the bifunctoriality of* ${}_x\circ_{\underline{x}}$:

   a) $1_f\,{}_x\circ_{\underline{x}}\,1_g = 1_{f\,{}_x\circ_{\underline{x}}\,g}$,

   b) $(\varphi_2\circ\varphi_1)\,{}_x\circ_{\underline{x}}\,(\psi_2\circ\psi_1) = (\varphi_2\,{}_x\circ_{\underline{x}}\,\psi_2)\circ(\varphi_1\,{}_x\circ_{\underline{x}}\,\psi_1)$,

3. *the naturality equations for β and γ:*

   a) $\beta^{x,\underline{x};y,\underline{y}}_{f_2,g_2,h_2}\circ((\varphi\,{}_x\circ_{\underline{x}}\,\phi)\,{}_y\circ_{\underline{y}}\,\psi) = (\varphi\,{}_x\circ_{\underline{x}}\,(\phi\,{}_y\circ_{\underline{y}}\,\psi))\circ\beta^{x,\underline{x};y,\underline{y}}_{f_1,g_1,h_1}$,

   b) $\gamma^{x,y}_{f_2,g_2}\circ(\varphi\,{}_x\circ_y\,\phi) = (\phi\,{}_y\circ_x\,\varphi)\circ\gamma^{x,y}_{f_1,g_1}$,

   where $\varphi : f_1\to f_2$, $\phi : g_1\to g_2$ *and* $\psi : h_1\to h_2$,

5. *the equations imposed by the functoriality of $\mathcal{C}$:*

a) $\mathcal{C}(1_X) = 1_{\mathcal{C}(X)}$,

b) $(f^\sigma)^\tau = f^{\sigma\circ\tau}$,

c) $(\varphi^\sigma)^\tau = \varphi^{\sigma\circ\tau}$

6. *the equations imposed by the functoriality of* $\mathcal{C}(\sigma)$*:*

   a) $1_f^\sigma = 1_{f^\sigma}$,

   b) $(\varphi \circ \psi)^\sigma = \varphi^\sigma \circ \psi^\sigma$.

Parallel associativity

We define a natural isomorphism ϑ, called *parallel associativity*, by taking

$$\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h} = \gamma^{\underline{x},x}_{g,f\,{}_y\circ_{\underline{y}}\,h} \circ \beta^{\underline{x},x;y,\underline{y}}_{g,f,h} \circ (\gamma^{x,\underline{x}}_{f,g}\,{}_y\circ_{\underline{y}}\,1_h) \,:\, (f\,{}_x\circ_{\underline{x}}\,g)\,{}_y\circ_{\underline{y}}\,h \longrightarrow (f\,{}_y\circ_{\underline{y}}\,h)\,{}_x\circ_{\underline{x}}\,g \tag{4.1.1}$$

for its components. The parallel associativity ϑ clearly represents the categorification of the homonymous law `(A2)`. Here are some first observations about ϑ.

Remark 4.5. *The natural isomorphism ϑ appears in* ($\beta\gamma$`-hexagon`) *and* ($\beta\gamma$`-decagon`).

1. *An isomorphism with the same source and target as $\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}$ could be introduced as the composition*

$$(\gamma^{\underline{y},y}_{h,f}\,{}_x\circ_{\underline{x}}\,1_g) \circ (\beta^{\underline{y},y;x,\underline{x}}_{h,f,g})^{-1} \circ \gamma^{y,\underline{y}}_{f\,{}_x\circ_{\underline{x}}\,g,h}$$

   which is as "natural" as the composition which we have fixed to be the definition of $\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}$. With this in mind, ($\beta\gamma$`-hexagon`) *can be read as: the two possible (and equally natural) definitions of $\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}$ are equal.*

2. *Also, ϑ appears twice in* ($\beta\gamma$`-decagon`)*, turning it into a hexagon by using explicilty the abbreviations $\vartheta^{y,\underline{y};z,\underline{z}}_{f\,{}_x\circ_{\underline{x}}\,g,h,k}$ (for the top horizontal sequence of arrows) and $1_{f}\,{}_x\circ_{\underline{x}}\,\vartheta^{y,\underline{y};z,\underline{z}}_{g,h,k}$ (for the bottom horizontal sequence of arrow).*

In the following two lemmas we show that ϑ is subject to certain *nice* (read: operadic) coherence conditions. We first show that the isomorphism $\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}$ has $\vartheta^{y,\underline{y};x,\underline{x}}_{f,h,g}$ as inverse.

Lemma 4.6. *The equality $\vartheta^{y,\underline{y};x,\underline{x}}_{f,h,g} \circ \vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h} = 1_{(f\,{}_x\circ_{\underline{x}}\,g)\,{}_y\circ_{\underline{y}}\,h}$ holds.*

Proof. The equality follows by the commutation of the (outer part of) diagram

$$\begin{tikzcd}
 & (g\,{}_{\underline{x}}\circ_x\,f)\,{}_y\circ_{\underline{y}}\,h \arrow[rr, "\beta^{\underline{x},x;y,\underline{y}}_{g,f,h}"] & & g\,{}_{\underline{x}}\circ_x\,(f\,{}_y\circ_{\underline{y}}\,h) \arrow[dr, "\gamma^{\underline{x},x}_{g,f\,{}_y\circ_{\underline{y}}\,h}"] & \\
(f\,{}_x\circ_{\underline{x}}\,g)\,{}_y\circ_{\underline{y}}\,h \arrow[ur, "\gamma^{x,\underline{x}}_{f,g}\,{}_y\circ_{\underline{y}}\,1_h"] & (g\,{}_{\underline{x}}\circ_x\,f)\,{}_y\circ_{\underline{y}}\,h \arrow[l, "\gamma^{\underline{x},x}_{g,f}\,{}_y\circ_{\underline{y}}\,1_h"'] & & g\,{}_{\underline{x}}\circ_x\,(f\,{}_y\circ_{\underline{y}}\,h) \arrow[ll, "(\beta^{\underline{x},x;y,\underline{y}}_{g,f,h})^{-1}"'] & (f\,{}_y\circ_{\underline{y}}\,h)\,{}_x\circ_{\underline{x}}\,g \arrow[l, "\gamma^{x,\underline{x}}_{f\,{}_y\circ_{\underline{y}}\,h,g}"'] \arrow[dl, "\gamma^{y,\underline{y}}_{f,h}\,{}_y\circ_{\underline{y}}\,1_g"] \\
 & h\,{}_{\underline{y}}\circ_y\,(f\,{}_x\circ_{\underline{x}}\,g) \arrow[ul, "\gamma^{\underline{y},y}_{h,f\,{}_x\circ_{\underline{x}}\,g}"] & & (h\,{}_{\underline{y}}\circ_y\,f)\,{}_x\circ_{\underline{x}}\,g \arrow[ll, "\beta^{\underline{y},y;x,\underline{x}}_{h,f,g}"'] &
\end{tikzcd}$$

in which the upper hexagon commutes by (γ`-involution`) and the lower hexagon commutes as an instance of ($\beta\gamma$`-hexagon`). ■

The following lemma shows two more laws satifsied by ϑ.

Lemma 4.7. *The following two equalities hold:*

- `(`$\beta\vartheta$`-pentagon)`

$$\vartheta^{y,\underline{y};x\underline{x}}_{f,h,g\,{}_{z}\circ_{\underline{z}}\,k}\circ\beta^{x,\underline{x};z,\underline{z}}_{f\,{}_{y}\circ_{\underline{y}}\,h,g,k}\circ(\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}\,{}_{z}\circ_{\underline{z}}\,1_k)=(\beta^{x,\underline{x};z,\underline{z}}_{f,g,k}\,{}_{y}\circ_{\underline{y}}\,1_h)\circ\vartheta^{y,\underline{y};z,\underline{z}}_{f\,{}_{x}\circ_{\underline{x}}\,g,h,k},$$

-`(`ϑ`-hexagon)`

$$\vartheta^{x,\underline{x};y,\underline{y}}_{f\,{}_{z}\circ_{\underline{z}}\,k,g,h}\circ(\vartheta^{x,\underline{x};z,\underline{z}}_{f,g,k}\,{}_{y}\circ_{\underline{y}}\,1_h)\circ\vartheta^{y,\underline{y};z,\underline{z}}_{f\,{}_{x}\circ_{\underline{x}}\,g,h,k}=(\vartheta^{y,\underline{y};z,\underline{z}}_{f,h,k}\,{}_{x}\circ_{\underline{x}}\,1_g)\circ\vartheta^{x,\underline{x};z,\underline{z}}_{f\,{}_{y}\circ_{\underline{y}}\,h,g,k}\circ(\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}\,{}_{z}\circ_{\underline{z}}\,1_k).$$

Proof. For the first equality, consider the diagram

$$\begin{tikzcd}
 & & (h\,{}_{\underline{y}}\circ_{y}\,(f\,{}_{x}\circ_{\underline{x}}\,g))\,{}_{z}\circ_{\underline{z}}\,k \arrow[ddll, "\beta^{\underline{y},y;x,\underline{x}}_{h,f,g}\,{}_{z}\circ_{\underline{z}}\,1_k"'] \arrow[ddrr, "\beta^{\underline{y},y;z,\underline{z}}_{h,f\,{}_{x}\circ_{\underline{x}}\,g,k}"] & & \\
 & & ((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k \arrow[u, "\gamma^{y,\underline{y}}_{f\,{}_{x}\circ_{\underline{x}}\,g,h}\,{}_{z}\circ_{\underline{z}}\,1_k"] \arrow[ddl, "\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}\,{}_{z}\circ_{\underline{z}}\,1_k"'] \arrow[ddr, "\vartheta^{y,\underline{y};z,\underline{z}}_{f\,{}_{x}\circ_{\underline{x}}\,g,h,k}"] & & \\
((h\,{}_{\underline{y}}\circ_{y}\,f)\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k \arrow[dddd, "\beta^{x,\underline{x};z,\underline{z}}_{h\,{}_{\underline{y}}\circ_{y}\,f,g,k}"'] & & & & h\,{}_{\underline{y}}\circ_{y}\,((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k) \arrow[dddd, "1_h\,{}_{\underline{y}}\circ_{y}\,\beta^{x,\underline{x};z,\underline{z}}_{f,g,k}"] \\
 & ((f\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k \arrow[ul, "(\gamma^{y,\underline{y}}_{f,h}\,{}_{x}\circ_{\underline{x}}\,1_g)\,{}_{z}\circ_{\underline{z}}\,1_k"] \arrow[dd, "\beta^{x,\underline{x};z,\underline{z}}_{f\,{}_{y}\circ_{\underline{y}}\,h,g,k}"'] & & ((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h \arrow[ur, "\gamma^{y,\underline{y}}_{(f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k,h}"'] \arrow[dd, "\beta^{x,\underline{x};z,\underline{z}}_{f,g,k}\,{}_{y}\circ_{\underline{y}}\,1_h"] & \\
 & & & & \\
 & (f\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{z}\circ_{\underline{z}}\,k) \arrow[rr, "\vartheta^{y,\underline{y};x\underline{x}}_{f,h,g\,{}_{z}\circ_{\underline{z}}\,k}"] \arrow[dl, "\gamma^{y,\underline{y}}_{h,f}\,{}_{x}\circ_{\underline{x}}\,1_{g\,{}_{z}\circ_{\underline{z}}\,k}"'] & & (f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{z}\circ_{\underline{z}}\,k))\,{}_{y}\circ_{\underline{y}}\,h & \\
(h\,{}_{\underline{y}}\circ_{y}\,f)\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{z}\circ_{\underline{z}}\,k) \arrow[rrrr, "\beta^{\underline{y},y;x,\underline{x}}_{h,f,g\,{}_{z}\circ_{\underline{z}}\,k}"] & & & & h\,{}_{\underline{y}}\circ_{y}\,(f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{z}\circ_{\underline{z}}\,k)) \arrow[ul, "\gamma^{\underline{y},y}_{h,f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{z}\circ_{\underline{z}}\,k)}"']
\end{tikzcd}$$

whose "inner" pentagon is `(`$\beta\vartheta$`-pentagon)` and whose "outer" pentagon commutes as an instance of `(`β`-pentagon)`. The equality follows by the commutations of all the diagrams "between" the two pentagons (two naturality squares for β and three squares expressing the definition of ϑ).

We have an analogous diagram for the second equality. The "inner" hexagon in the diagram

$$\begin{tikzcd}
((g\,{}_{\underline{x}}\circ_{x}\,f)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k \arrow[rrr, "\beta^{\underline{x},x;y,\underline{y}}_{g,f,h}\,{}_{z}\circ_{\underline{z}}\,1_k"] \arrow[dd, "\vartheta^{y,\underline{y};z,\underline{z}}_{g\,{}_{\underline{x}}\circ_{x}\,f,h,k}"'] & & & (g\,{}_{\underline{x}}\circ_{x}\,(f\,{}_{y}\circ_{\underline{y}}\,h))\,{}_{z}\circ_{\underline{z}}\,k \arrow[dl, "\gamma^{\underline{x},x}_{g,f\,{}_{y}\circ_{\underline{y}}\,h}\,{}_{z}\circ_{\underline{z}}\,1_k"'] \arrow[dd, "\beta^{\underline{x},x;z,\underline{z}}_{g,f\,{}_{y}\circ_{\underline{y}}\,h,k}"] \\
 & ((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k \arrow[ul, "(\gamma^{x,\underline{x}}_{f,g}\,{}_{y}\circ_{\underline{y}}\,1_h)\,{}_{z}\circ_{\underline{z}}\,1_k"'] \arrow[r, "\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}\,{}_{z}\circ_{\underline{z}}\,1_k"] \arrow[d, "\vartheta^{y,\underline{y};z,\underline{z}}_{f\,{}_{x}\circ_{\underline{x}}\,g,h,k}"'] & ((f\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k \arrow[d, "\vartheta^{x,\underline{x};z,\underline{z}}_{f\,{}_{y}\circ_{\underline{y}}\,h,g,k}"] & \\
((g\,{}_{\underline{x}}\circ_{x}\,f)\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h \arrow[dd, "\beta^{\underline{x},x;z,\underline{z}}_{g,f,k}\,{}_{y}\circ_{\underline{y}}\,1_h"'] & ((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h \arrow[l, "(\gamma^{x,\underline{x}}_{f,g}\,{}_{z}\circ_{\underline{z}}\,1_k)\,{}_{y}\circ_{\underline{y}}\,1_h"'] \arrow[d, "\vartheta^{x,\underline{x};z,\underline{z}}_{f,g,k}\,{}_{y}\circ_{\underline{y}}\,1_h"'] & ((f\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{x}\circ_{\underline{x}}\,g \arrow[r, "\gamma^{x,\underline{x}}_{(f\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k,g}"] \arrow[d, "\vartheta^{y,\underline{y};z,\underline{z}}_{f,h,k}\,{}_{x}\circ_{\underline{x}}\,1_g"] & g\,{}_{\underline{x}}\circ_{x}\,((f\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k) \arrow[dd, "1_g\,{}_{\underline{x}}\circ_{x}\,\vartheta^{y,\underline{y};z,\underline{z}}_{f,h,k}"] \\
 & ((f\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h \arrow[r, "\vartheta^{x,\underline{x};y,\underline{y}}_{f\,{}_{z}\circ_{\underline{z}}\,k,g,h}"] & ((f\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{x}\circ_{\underline{x}}\,g \arrow[dr, "\gamma^{x,\underline{x}}_{(f\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h,g}"] & \\
(g\,{}_{\underline{x}}\circ_{x}\,(f\,{}_{z}\circ_{\underline{z}}\,k))\,{}_{y}\circ_{\underline{y}}\,h \arrow[ur, "\gamma^{\underline{x},x}_{g,f\,{}_{z}\circ_{\underline{z}}\,k}\,{}_{y}\circ_{\underline{y}}\,1_h"] \arrow[rrr, "\beta^{\underline{x},x;y,\underline{y}}_{g,f\,{}_{z}\circ_{\underline{z}}\,k,h}"] & & & g\,{}_{\underline{x}}\circ_{x}\,((f\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h)
\end{tikzcd}$$

is (ϑ-hexagon) from the claim, and the "outer" hexagon is an instance of ($\beta\gamma$-decagon), and the equality follows by the commutations of all the diagrams "between" the two hexagons (these are the four naturality squares for ϑ and two squares which express the definition of ϑ). ■

4.1.2 Canonical diagrams and the coherence theorem

The coherence theorem that we shall prove has the form: *all diagrams of canonical arrows commute in* $\mathcal{C}(X)$. In order to formulate it rigorously, we shall first specify what a diagram of canonical arrows is exactly. Denoting with $\underline{\mathcal{C}}$ the underlying functor of $\mathcal{C}$, in this part we essentially introduce a syntax for the free categorified entries-only cyclic operad without units generated by $\underline{\mathcal{C}}$. However, since the purpose of the syntax is solely to distinguish the canonical arrows of $\mathcal{C}(X)$, the formalism will be left without any equations.

The syntax $\mathtt{Free}_{\underline{\mathcal{C}}}$

Let $P_{\underline{\mathcal{C}}}$ be as in (2.2.2) and let Σ range over bijections of finite sets. Recall that $\mathtt{V}$ is the set of variables whose existence is assumed throughout the thesis.

The syntax $\mathtt{Free}_{\underline{\mathcal{C}}}$ *of canonical diagrams (or, of* $\beta\gamma\sigma$*-arrows) of* $\mathcal{C}$, contains two kinds of typed expressions, the *object terms* and the *arrow terms* (as all the other formal systems that we shall introduce in the remaining of the section).

The syntax of object terms is obtained from raw (i.e. not *yet* typed) object terms

$$\boxed{\mathcal{W} ::= \underline{a} \mid \mathcal{W}\,{}_x\Box_y\,\mathcal{W} \mid \mathcal{W}^{\sigma}}$$

where $a \in P_{\underline{\mathcal{C}}}$, $x, y \in \mathtt{V}$, and $\sigma \in \Sigma$, by typing them as $\mathcal{W} : X$, where X ranges over finite sets. The assignment of types is done by the following rules:

$$\frac{a \in \underline{\mathcal{C}}(X)}{\underline{a} : X} \qquad \frac{\mathcal{W}_1 : X \quad \mathcal{W}_2 : Y \quad x \in X \;\; y \in Y \quad X\backslash\{x\} \cap Y\backslash\{y\} = \emptyset}{\mathcal{W}_1 \,{}_x\Box_y\, \mathcal{W}_2 : X\backslash\{x\} \cup Y\backslash\{y\}} \qquad \frac{\mathcal{W} : X \quad \sigma : Y \to X}{\mathcal{W}^{\sigma} : Y}$$

Observe that the object terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$ are exactly the combinators of §2.2, except that they do not incorporate terms $id_{x,y}$ that encode units of (categorified) cyclic operads.

Remark 4.8. *The notation* ${}_x\Box_y$ *(rather then* ${}_x\circ_y$ *) for the syntax of partial composition operations is chosen merely to avoid confusion with the symbol* $\circ$*, used to denote the (usual) composition of morphisms in a category.*

To the syntax of object terms we add the syntax of arrow terms, obtained from raw arrow terms

$$\boxed{\Phi ::= \begin{cases} 1_{\mathcal{W}} \mid \beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \mid \beta^{x,\underline{x};y,\underline{y}\;\,-1}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \mid \gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2} \\ \varepsilon1^{\sigma}_{\underline{a}} \mid \varepsilon1^{\sigma\;-1}_{\underline{a}} \mid \varepsilon2_{\mathcal{W}} \mid \varepsilon2^{-1}_{\mathcal{W}} \mid \varepsilon3^{\sigma,\tau}_{\mathcal{W}} \mid \varepsilon3^{\sigma,\tau\;-1}_{\mathcal{W}} \mid \varepsilon4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} \mid \varepsilon4^{x,y;x',y'\;-1}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} \\ \Phi \circ \Phi \mid \Phi \,{}_x\Box_y\, \Phi \mid \Phi^{\sigma}, \end{cases}}$$

by assigning them types in the form of ordered pairs $(\mathcal{W}_1, \mathcal{W}_2)$ of object terms, denoted by $\mathcal{W}_1 \to \mathcal{W}_2$, as shown in Figure 4.1, where it is also (implicitly) assumed that all the object terms that appear in the types of the arrow terms are well-formed. Given an arrow term $\Phi : \mathcal{W}_1 \to \mathcal{W}_2$, we call the object term $\mathcal{W}_1$ *the source of* Φ and the object term $\mathcal{W}_2$ *the target of* Φ.

Remark 4.9. *Observe that, for all well-typed arrow terms* $\Phi : \mathcal{W}_1 \to \mathcal{W}_2$ *of* $\mathtt{Free}_{\underline{\mathcal{C}}}$*, the object terms* $\mathcal{W}_1$ *and* $\mathcal{W}_2$ *have the same type.*

$$\frac{}{1_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}}$$

$$\frac{}{\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} : (\mathcal{W}_1{}_x\square_{\underline{x}}\mathcal{W}_2)_y\square_{\underline{y}}\mathcal{W}_3 \to \mathcal{W}_1{}_x\square_{\underline{x}}(\mathcal{W}_2{}_y\square_{\underline{y}}\mathcal{W}_3)}$$

$$\frac{}{\beta^{x,\underline{x};y,\underline{y}\ -1}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} : \mathcal{W}_1{}_x\square_{\underline{x}}(\mathcal{W}_2{}_y\square_{\underline{y}}\mathcal{W}_3) \to (\mathcal{W}_1{}_x\square_{\underline{x}}\mathcal{W}_2)_y\square_{\underline{y}}\mathcal{W}_3}$$

$$\frac{}{\gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2} : \mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2 \to \mathcal{W}_2\,{}_y\square_x\,\mathcal{W}_1}$$

$$\frac{}{\varepsilon 1^{\sigma}_{\underline{a}} : \underline{a}^{\sigma} \to \underline{a^{\sigma}}} \qquad \frac{}{\varepsilon 1^{\sigma\ -1}_{\underline{a}} : \underline{a^{\sigma}} \to \underline{a}^{\sigma}}$$

$$\frac{}{\varepsilon 2_{\mathcal{W}} : \mathcal{W}^{id_X} \to \mathcal{W}} \qquad \frac{}{\varepsilon 2^{-1}_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}^{id_X}}$$

$$\frac{}{\varepsilon 3^{\sigma,\tau}_{\mathcal{W}} : (\mathcal{W}^{\sigma})^{\tau} \to \mathcal{W}^{\sigma\circ\tau}} \qquad \frac{}{\varepsilon 3^{\sigma,\tau\ -1}_{\mathcal{W}} : \mathcal{W}^{\sigma\circ\tau} \to (\mathcal{W}^{\sigma})^{\tau}}$$

$$\frac{\begin{array}{c}\sigma : Z \to X\backslash\{x\}\cup Y\backslash\{y\}\\ \sigma_1 : \sigma^{-1}[X\backslash\{x\}]\cup\{x'\} \to X \quad \sigma_1|^{X\backslash\{x\}} = \sigma|^{X\backslash\{x\}} \quad \sigma_1(x') = x\\ \sigma_2 : \sigma^{-1}[Y\backslash\{y\}]\cup\{y'\} \to Y \quad \sigma_2|^{Y\backslash\{y\}} = \sigma|^{Y\backslash\{y\}} \quad \sigma_2(y') = y\end{array}}{\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} : (\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^{\sigma} \to \mathcal{W}_1^{\sigma_1}\,{}_{x'}\square_{y'}\,\mathcal{W}_2^{\sigma_2}}$$

$$\frac{\begin{array}{c}\sigma_1 : X' \to X \quad \sigma_1(x') = x\\ \sigma_2 : Y' \to Y \quad \sigma_2(y') = y\\ \sigma : X'\backslash\{x'\}\cup Y'\backslash\{y'\} \to X\backslash\{x\}\cup Y\backslash\{y\} \quad \sigma = \sigma_1|^{X'\backslash\{x'\}}\cup\sigma_2|^{Y'\backslash\{y'\}}\end{array}}{\varepsilon 4^{x,y;x',y'\ -1}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} : \mathcal{W}_1^{\sigma_1}\,{}_{x'}\square_{y'}\,\mathcal{W}_2^{\sigma_2} \to (\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^{\sigma}}$$

$$\frac{\Phi_1 : \mathcal{W}_1 \to \mathcal{W}_2 \quad \Phi_2 : \mathcal{W}_2 \to \mathcal{W}_3}{\Phi_2 \circ \Phi_1 : \mathcal{W}_1 \to \mathcal{W}_3} \qquad \frac{\Phi_1 : \mathcal{W}_1 \to \mathcal{W}_1' \quad \Phi_2 : \mathcal{W}_2 \to \mathcal{W}_2'}{\Phi_1\,{}_x\square_y\,\Phi_2 : \mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2 \to \mathcal{W}_1'\,{}_x\square_y\,\mathcal{W}_2'}$$

$$\frac{\Phi : \mathcal{W}_1 \to \mathcal{W}_2 \quad \sigma : Y \to X}{\Phi^{\sigma} : \mathcal{W}_1^{\sigma} \to \mathcal{W}_2^{\sigma}}$$

FIGURE 4.1: Typing rules for the arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$

The collection of object terms of type X, together with the collection of arrow terms whose source and target have type X, will be denoted by $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$.

The interpretation of $\mathtt{Free}_{\underline{\mathcal{C}}}$ in $\mathcal{C}$

The semantics of $\mathtt{Free}_{\underline{\mathcal{C}}}$ in $\mathcal{C}$ is what distinguishes canonical arrows (or $\beta\gamma\sigma$-arrows) of $\mathcal{C}(X)$: they will be precisely the interpretations of the arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$. Given that the axiom (EQ) remains strict in the transition from Definition 1.4 to Definition 4.1, the interpretations of the arrow terms whose denotations contain the symbol ε (and which all encode the properties of the action of the symmetric group) will be identities.

The interpretation function $[[-]]_X : \mathtt{Free}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$ is defined recursively, as follows:

$$[[\underline{a}]]_X = a, \qquad [[\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2]]_X = [[\mathcal{W}_1]]_{X_1}\,{}_x\circ_y\,[[\mathcal{W}_2]]_{X_2}, \qquad [[\mathcal{W}^{\sigma}]]_X = ([[\mathcal{W}]]_Y)^{\sigma},$$

and

- ◇ $[[1_{\mathcal{W}}]]_X = 1_{[[\mathcal{W}]]_X}$,
- ◇ $[[\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3}]]_X = \beta^{x,\underline{x};y,\underline{y}}_{[[\mathcal{W}_1]]_{X_1},[[\mathcal{W}_2]]_{X_2},[[\mathcal{W}_3]]_{X_3}}$,

- $\diamond$ $[[{\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3}}^{-1}]]_X = {\beta^{x,\underline{x};y,\underline{y}}_{[[\mathcal{W}_1]]_{X_1},[[\mathcal{W}_2]]_{X_2},[[\mathcal{W}_3]]_{X_3}}}^{-1}$,
- $\diamond$ $[[\gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2}]]_X = \gamma^{x,y}_{[[\mathcal{W}_1]]_{X_1},[[\mathcal{W}_2]]_{X_2}}$,
- $\diamond$ $[[\varepsilon 1^{\sigma}_{\underline{a}}]]_X = 1_{[[\underline{a}^{\sigma}]]_X}$, $[[{\varepsilon 1^{\sigma}_{\underline{a}}}^{-1}]]_X = 1_{[[\underline{a}^{\sigma}]]_X}$,
- $\diamond$ $[[\varepsilon 2_{\mathcal{W}}]]_X = 1_{[[\mathcal{W}^{id_X}]]_X}$, $[[\varepsilon 2^{-1}_{\mathcal{W}}]]_X = 1_{[[\mathcal{W}]]_X}$,
- $\diamond$ $[[\varepsilon 3^{\sigma,\tau}_{\mathcal{W}}]]_X = 1_{[[(\mathcal{W}^{\sigma})^{\tau}]]_X}$, $[[{\varepsilon 3^{\sigma,\tau}_{\mathcal{W}}}^{-1}]]_X = 1_{[[\mathcal{W}^{\sigma\circ\tau}]]_X}$
- $\diamond$ $[[\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma}]]_X = 1_{[[(\mathcal{W}_1{}_x\square_y\mathcal{W}_2)^{\sigma}]]_X}$, $[[{\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma}}^{-1}]]_X = 1_{[[\mathcal{W}_1^{\sigma_1}{}_{x'}\square_{y'}\mathcal{W}_2^{\sigma_2}]]_X}$,
- $\diamond$ $[[\Phi_2\circ\Phi_1]]_X = [[\Phi_2]]_X\circ[[\Phi_1]]_X$,
- $\diamond$ $[[\Phi_1{}_x\square_y\Phi_2]]_X = [[\Phi_1]]_{X_1}{}_x\circ_y[[\Phi_2]]_{X_2}$, and
- $\diamond$ $[[\Phi^{\sigma}]]_X = ([[\Phi]]_Y)^{\sigma}$.

Lemma 4.10. *The interpretation function $[[-]]_X : \mathtt{Free}_{\underline{\mathcal{C}}}(X)\to\mathcal{C}(X)$ is well-defined, in the sense that, for an arrow term $\Phi:\mathcal{W}_1\to\mathcal{W}_2$ of $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$, we have that*

$$[[\Phi]]_X : [[\mathcal{W}_1]]_X \to [[\mathcal{W}_2]]_X.$$

Proof. The proof goes by easy structural induction on the (object and arrow) terms of $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$. For arrow terms whose denotations contain ε, the claim holds thanks to the equivariance axiom `(EQ)` for $\mathcal{C}$. We show this only for the arrow term

$$\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} : (\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^{\sigma}\to\mathcal{W}_1^{\sigma_1}\,{}_{x'}\square_{y'}\,\mathcal{W}_2^{\sigma_2},$$

where $\mathcal{W}_1 : X$, $\mathcal{W}_2 : Y$, and σ, σ_1 and σ_2 are as in the appropriate typing rule of Figure 4.1. Denote $U = X\backslash\{x\}\cup Y\backslash\{y\}$. We then have

$$[[\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma}]]_U = 1_{[[(\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^{\sigma}]]_U} : [[(\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^{\sigma}]]_U \to [[(\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^{\sigma}]]_U$$

and, by the axiom `(EQ)` for $\mathcal{C}$, we have

$$\begin{aligned}
[[(\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^{\sigma}]]_Z &= ([[\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2]]_U)^{\sigma}\\
&= ([[\mathcal{W}_1]]_X\,{}_x\circ_y\,[[\mathcal{W}_2]]_Y)^{\sigma}\\
&= ([[\mathcal{W}_1]]_X)^{\sigma_1}\,{}_{x'}\circ_{y'}\,([[\mathcal{W}_2]]_Y)^{\sigma_2}\\
&= [[\mathcal{W}_1^{\sigma_1}]]_{\sigma^{-1}[X\backslash\{x\}]\cup\{x'\}}\,{}_{x'}\circ_{y'}\,[[\mathcal{W}_2^{\sigma_2}]]_{\sigma^{-1}[Y\backslash\{y\}]\cup\{y'\}}\\
&= [[\mathcal{W}_1^{\sigma_1}{}_{x'}\square_{y'}\mathcal{W}_2^{\sigma_2}]]_{\sigma^{-1}[X\backslash\{x\}]\cup\sigma^{-1}[Y\backslash\{y\}]}\\
&= [[\mathcal{W}_1^{\sigma_1}{}_{x'}\square_{y'}\mathcal{W}_2^{\sigma_2}]]_Z.
\end{aligned}$$

■

Relying on Lemma 4.10, we define a *canonical diagram in $\mathcal{C}(X)$ as a pair of parallel morphisms (i.e. morphisms that share the same source and target) arising as interpretations of two arrow terms of the same type of* $\mathtt{Free}_{\underline{\mathcal{C}}}$.

The coherence theorem

We can now state precisely the coherence theorem for $\mathcal{C}$.

Coherence Theorem. *For any finite set X and for any pair of arrow terms $\Phi,\Psi : \mathcal{W}_1\to\mathcal{W}_2$ of the same type in $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$, the equality $[[\Phi]]_X = [[\Psi]]_X$ holds in $\mathcal{C}(X)$.*

In the remainder of Section 4.1, we prove the coherence theorem.

4.1.3 The first reduction: getting rid of symmetries

Intuitively, the first reduction cuts down the coherence problem of $\mathcal{C}$ to the problem of commutation of all *diagrams of $\beta\gamma$-arrows* of $\mathcal{C}$. We introduce first the syntax of these diagrams.

The syntax $\underline{\texttt{Free}_{\mathcal{C}}}$

The syntax that we are about to introduce is obtained by removing the term constructor $(-)^{\sigma}$ from the list of raw object and raw arrow terms of $\texttt{Free}_{\underline{\mathcal{C}}}$, as well as all the arrow terms of $\texttt{Free}_{\underline{\mathcal{C}}}$ whose denotation contains ε.

The $\beta\gamma$*-reduction* of $\texttt{Free}_{\underline{\mathcal{C}}}$, denoted by $\underline{\texttt{Free}_{\mathcal{C}}}$, is specified as follows. The collection of object terms of $\underline{\texttt{Free}_{\mathcal{C}}}$ is determined by raw object terms

$$\boxed{W ::= \underline{a} \,|\, W \;{}_x\square_y\, W}$$

where $a \in P_{\underline{\mathcal{C}}}$ and $x, y \in \texttt{V}$, and typing rules

$$\boxed{\frac{a \in \underline{\mathcal{C}}(X)}{\underline{a} : X} \qquad \frac{W_1 : X \quad W_2 : Y \quad x \in X \;\; y \in Y \quad (X\backslash\{x\}) \cap (Y\backslash\{y\}) = \emptyset}{W_1 \;{}_x\square_y\, W_2 : X\backslash\{x\} \cup Y\backslash\{y\}}}$$

The collection of arrow terms of $\underline{\texttt{Free}_{\mathcal{C}}}$ is obtained from the raw arrow terms

$$\boxed{\varphi ::= 1_W \;|\; \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3} \;|\; {\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}}^{-1} \;|\; \gamma^{x,y}_{W_1,W_2} \;|\; \varphi \circ \varphi \;|\; \varphi \;{}_x\square_y\, \varphi}$$

by typing them with pairs of object terms as shown in Figure 4.2.

$$\frac{}{1_W : W \to W}$$

$$\frac{}{\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3} : (W_1{}_x\square_{\underline{x}}W_2)_y\square_{\underline{y}}W_3 \to W_1{}_x\square_{\underline{x}}(W_2{}_y\square_{\underline{y}}W_3)}$$

$$\frac{}{{\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}}^{-1} : W_1{}_x\square_{\underline{x}}(W_2{}_y\square_{\underline{y}}W_3) \to (W_1{}_x\square_{\underline{x}}W_2)_y\square_{\underline{y}}W_3}$$

$$\frac{}{\gamma^{x,y}_{W_1,W_2} : W_1{}_x\square_y W_2 \to W_2\;{}_y\square_x W_1}$$

$$\frac{\varphi_1 : W_1 \to W_2 \quad \varphi_2 : W_2 \to W_3}{\varphi_2 \circ \varphi_1 : W_1 \to W_3} \qquad \frac{\varphi_1 : W_1 \to W_1' \quad \varphi_2 : W_2 \to W_2'}{\varphi_1 \;{}_x\square_y\, \varphi_2 : W_1 \;{}_x\square_y\, W_2 \to W_1' \;{}_x\square_y\, W_2'}$$

FIGURE 4.2: Typing rules for the arrow terms of $\underline{\texttt{Free}_{\mathcal{C}}}$

Analogously as before, we shall denote the collection of object terms of type X, together with the collection of arrow terms whose source and target are object terms of type X, by $\underline{\texttt{Free}_{\mathcal{C}}}(X)$.

Remark 4.11. *Notice that the type of an arrow term φ of $\underline{\texttt{Free}_{\mathcal{C}}}$ is determined completely by φ only, that is, by the* indices of φ *and their* order of appearance in φ. *This allows us to write $W_s(\varphi)$ and $W_t(\varphi)$ for the source and target of φ, respectively.*

Furthermore, observe that, for an arbitrary arrow term $\varphi : W_1 \to W_2$, the parameters and variables that appear in W_1 are exactly *the parameters and variables that appear in W_2.*

The interpretation of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$ in $\mathcal{C}$

The semantics of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$ in $\mathcal{C}$ is what distinguishes *$\beta\gamma$-arrows* of $\mathcal{C}(X)$ from all other canonical arrows of $\mathcal{C}(X)$.

The interpretation function $[-]_X : \underline{\texttt{Free}}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$ is defined recursively, as follows:

$$[\underline{a}]_X = a, \qquad [W_1 \, {}_x\Box_y \, W_2]_X = [W_1]_{X_1} \, {}_x\circ_y \, [W_2]_{X_2},$$

and

- $\diamond$ $[1_W]_X = 1_{[W]_X}$,
- $\diamond$ $[\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}]_X = \beta^{x,\underline{x};y,\underline{y}}_{[W_1]_{X_1},[W_2]_{X_2},[W_3]_{X_3}}$,
- $\diamond$ $[\beta^{x,\underline{x};y,\underline{y}\ \ -1}_{W_1,W_2,W_3}]_X = \beta^{x,\underline{x};y,\underline{y}\ \ -1}_{[W_1]_{X_1},[W_2]_{X_2},[W_3]_{X_3}}$,
- $\diamond$ $[\gamma^{x,y}_{W_1,W_2}]_X = \gamma^{x,y}_{[W_1]_{X_1},[W_2]_{X_2}}$,
- $\diamond$ $[\varphi_2 \circ \varphi_1]_X = [\varphi_2]_X \circ [\varphi_1]_X$, and
- $\diamond$ $[\varphi_1 \, {}_x\Box_y \, \varphi_2]_X = [\varphi_1]_{X_1} \, {}_x\circ_y \, [\varphi_2]_{X_2}$.

Lemma 4.12. *The interpretation function $[-]_X : \underline{\texttt{Free}}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$ is well-defined, in the sense that, for an arrow term $\varphi : W_1 \to W_2$ of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}(X)$, we have that*

$$[\varphi]_X : [W_1]_X \to [W_2]_X.$$

Proof. This is a direct consequence of Lemma 4.10. ∎

An auxiliary typing system for the raw arrow terms of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$

In this part, we introduce a slightly more permissive typing system for the raw arrow terms of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$, by "relaxing" the rule for typing the composition $\varphi_2 \circ \varphi_1$. More precisely, the new formal system, which we shall denote with $\utilde{\texttt{Free}}_{\underline{\mathcal{C}}}$, will be the same as $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$, except for the composition rule for arrow terms, where we add a degree of freedom by allowing the composition not only "along" the *same* typed object term, but also "along" the *α-equivalent* ones.

In order to define α-equivalence on object terms of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$, we introduce some terminology. For a parameter $a \in \underline{\mathcal{C}}(X)$ of $P_{\underline{\mathcal{C}}}$, we say that X is the set of *free variables* of a, and we write $FV(a) = X$. For an object term $W : Y$, we shall denote with $P_{\underline{\mathcal{C}}}(W)$ the set of all parameters of $P_{\underline{\mathcal{C}}}$ that appear in W. The α-equivalence on object terms of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$ is the smallest equivalence relation $\equiv$ generated by the rule

$$\frac{\begin{array}{c} W_1{:}X \quad W_2{:}Y \quad x{\in}X \quad y{\in}Y \quad X\backslash\{x\}{\cap}Y\backslash\{y\}{=}\emptyset \quad x',y'{\notin}X\backslash\{x\}{\cup}Y\backslash\{y\} \quad x'{\neq}y' \\ a{\in}P_{\underline{\mathcal{C}}}(W_1) \quad FV(a){=}X_1 \quad x{\in}X_1{\cap}X \\ b{\in}P_{\underline{\mathcal{C}}}(W_2) \quad FV(b){=}Y_1 \quad y{\in}Y_1{\cap}Y \\ \tau_1{:}X_1\backslash\{x\}{\cup}\{x'\}{\to}X_1 \quad \tau_1|_{X_1\backslash\{x\}}{=}id_{X_1\backslash\{x\}} \quad \tau_1(x'){=}x \\ \tau_2{:}Y_1\backslash\{y\}{\cup}\{y'\}{\to}Y_1 \quad \tau_2|_{Y_1\backslash\{y\}}{=}id_{Y_1\backslash\{y\}} \quad \tau_2(y'){=}y \end{array}}{W_1 \, {}_x\Box_y \, W_2 \equiv W_1[\underline{a^{\tau_1}}/\underline{a}] \, {}_{x'}\Box_{y'} \, W_2[\underline{b^{\tau_2}}/\underline{b}]} \tag{4.1.2}$$

where $W_1[\underline{a^{\tau_1}}/\underline{a}]$ (resp. $W_2[\underline{b^{\tau_2}}/\underline{b}]$) denotes the result of the substitution of the parameter $\underline{a^{\tau_1}}$ (resp. $\underline{b^{\tau_2}}$) for the parameter $\underline{a}$ (resp. $\underline{b}$) in W_1 (resp. W_2), which is, moreover, congruent with respect to ${}_x\Box_y$. The intuition is simpler than it might seem: the rule defining $\equiv$ formalises a particular case of equivariance on objects (see Remark 4.3). Here is an example.

EXAMPLE 4.13. Returning to the syntax $\texttt{Free}_{\underline{\mathcal{C}}}$, which encompases terms of the form $\mathcal{W}^{\sigma}$, observe that, fixing $\sigma = id_{X\backslash\{x\}\cup Y\backslash\{y\}}$, by (EQ), we have

$$\begin{aligned}
[[\underline{a}\,{}_{x}\square_{y}\,\underline{b}]]_{X\backslash\{x\}\cup Y\backslash\{y\}} &= ([[\underline{a}]]_X\,{}_{x}\square_{y}\,[[\underline{b}]]_Y)^{id_{X\backslash\{x\}\cup Y\backslash\{y\}}}_{X\backslash\{x\}\cup Y\backslash\{y\}}\\
&= [[\underline{a}]]^{\tau_1}_{X\backslash\{x\}\cup\{x'\}}\,{}_{x'}\square_{y'}\,[[\underline{b}]]^{\tau_2}_{Y\backslash\{y\}\cup\{y'\}}\\
&= [[\underline{a^{\tau_1}}]]_{X\backslash\{x\}\cup\{x'\}}\,{}_{x'}\square_{y'}\,[[\underline{b^{\tau_2}}]]_{Y\backslash\{y\}\cup\{y'\}}\\
&= [[\underline{a^{\tau_1}\,{}_{x'}\square_{y'}\,b^{\tau_2}}]]_{X\backslash\{x\}\cup Y\backslash\{y\}}.
\end{aligned}$$

The first and the last object term in this sequence of equalities of interpretations are object terms of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$ and they are α-equivalent. From the tree-wise perspective, the equivalence relation $\equiv$ corresponds exactly to the α-equivalence of unrooted trees (see §2.3.1).

The substitution of parameters of object terms canonically induces substitution of parameters of arrow terms of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$. For an arrow term $\varphi : W_1 \to W_2$ of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$, $a \in P_{\underline{\mathcal{C}}}(U)$ and $a^{\tau} \notin P_{\underline{\mathcal{C}}}(U)$, such that $W_1[\underline{a^{\tau}}/\underline{a}]$ (and thus also $W_2[\underline{a^{\tau}}/\underline{a}]$) is well-typed, the arrow term $\varphi[\underline{a^{\tau}}/\underline{a}] : W_1[\underline{a^{\tau}}/\underline{a}] \to W_2[\underline{a^{\tau}}/\underline{a}]$ is defined straightforwardly by modifying the indices of φ as dictated by the substitution $W_1[\underline{a^{\tau}}/\underline{a}]$.

EXAMPLE 4.14. If $\varphi = \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}$, where $x \in X_1$, $a \in \underline{\mathcal{C}}(X_1)$ and $a \in P_{\underline{\mathcal{C}}}(W_1)$, then

$$\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}[\underline{a^{\tau}}/\underline{a}] = \beta^{x',\underline{x};y,\underline{y}}_{W_1[\underline{a^{\tau}}/\underline{a}],W_2,W_3},$$

where $x' = \tau^{-1}(x)$. $\square$

We shall need the following property of the "interpretation of substitution".

Lemma 4.15. *Let W be an object term of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}(X)$ and let $x \in X$. Let $a \in P_{\underline{\mathcal{C}}}(W)$ be such that $x \in FV(a)$, and suppose that $\tau : FV(a)\backslash\{x\} \cup \{x'\} \to FV(a)$ renames x to x'. We then have*

$$[\,W[\underline{a^{\tau}}/\underline{a}]\,] = [\,W\,]^{\sigma},$$

where $\sigma : X\backslash\{x\} \cup \{x'\} \to X$ renames x to x'. Additionally, for any arrow term φ of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}(X)$ such that $W_s(\varphi) = W$, we have

$$[\,\varphi[\underline{a^{\tau}}/\underline{a}]\,] = [\,\varphi\,]^{\sigma}.$$

Proof. By easy structural induction, thanks to (EQ), ($\beta\sigma$), ($\gamma\sigma$), (EQ-mor) and Remark 4.4.6. ■

Lemma 4.16. *If $W_1 \equiv W_2$, then $[W_1]_X = [W_2]_X$.*

Proof. By induction on the proof of $W_1 \equiv W_2$ and Lemma 4.15. ■

We now specify the syntax $\underset{\sim}{\texttt{Free}}_{\underline{\mathcal{C}}}$. The object terms and the raw arrow terms of $\underset{\sim}{\texttt{Free}}_{\underline{\mathcal{C}}}$ are exactly the object terms and the raw arrow terms of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$. The type of an arrow term φ of $\underset{\sim}{\texttt{Free}}_{\underline{\mathcal{C}}}$ is again a pair of object terms, which we shall now denote with $\vdash \varphi : U \to V$. The typing rules for arrow terms are the same as the typing rules for arrow terms of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$, given in Figure 4.2, except for the composition rule, for which we now take the rule given in Figure 4.3.

$$\frac{\vdash \varphi_1 : W_1 \to W_2 \qquad \vdash \varphi_2 : W_2' \to W_3 \qquad W_2 \equiv W_2'}{\vdash \varphi_2 \circ \varphi_1 : W_1 \to W_3}$$

FIGURE 4.3: Typing rule for the composition of arrow terms in $\underset{\sim}{\texttt{Free}}_{\underline{\mathcal{C}}}$

As usual, $\underset{\sim}{\texttt{Free}}_{\underline{\mathcal{C}}}(X)$ shall denote the collection of object terms of type X, together with the collection of arrow terms whose source and target are objects terms of type X, of $\underset{\sim}{\texttt{Free}}_{\underline{\mathcal{C}}}$.

The interpretation of $\utilde{\mathtt{Free}_{\mathcal{C}}}(X)$ in $\mathcal{C}(X)$, is defined (and denoted) exactly as the interpretation $[-]_X$. In particular, the interpretation of the "relaxed" composition is defined by $[\varphi_2 \circ \varphi_1]_X = [\varphi_2]_X \circ [\varphi_1]_X$. The following lemma is a direct consequence of Lemma 4.16.

Lemma 4.17. *The interpretation function* $[-]_X : \utilde{\mathtt{Free}_{\mathcal{C}}}(X) \to \mathcal{C}(X)$ *is well-defined.*

The purpose of the syntax $\utilde{\mathtt{Free}_{\mathcal{C}}}$ in the proof of the coherence theorem is to be a natural "intermedium" in the transition from $\mathtt{Free}_{\underline{\mathcal{C}}}$ to $\underline{\mathtt{Free}}_{\mathcal{C}}$. As we shall see, $\utilde{\mathtt{Free}_{\mathcal{C}}}$ is more suitable as an immediate environment for the removal symmetries from $\mathtt{Free}_{\underline{\mathcal{C}}}$. The following lemma then enables us to complete the transition to $\underline{\mathtt{Free}}_{\mathcal{C}}$.

Lemma 4.18. *If* $\vdash \varphi : U \to V$ *is an arrow term of* $\utilde{\mathtt{Free}_{\mathcal{C}}}(X)$ *and if* $U \equiv U'$, *then there exists an arrow term* $\varphi^{U'} : U' \to W_t(\varphi^{U'})$ *of* $\underline{\mathtt{Free}}_{\mathcal{C}}(X)$, *such that*

$$W_t(\varphi^{U'}) \equiv V \qquad \textit{and} \qquad [\varphi]_X = [\varphi^{U'}]_X.$$

Proof. By induction on the structure of φ.

- If $\varphi = 1_U$, then $\varphi^{U'} = 1_{U'}$. We conclude by (EQ) and Remark 4.4.6(a), for $\sigma = id_X$.
- Suppose that $\varphi = \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}$. The source of φ is then $U = (W_1\, {}_x\square_{\underline{x}}\, W_2)\, {}_y\square_{\underline{y}}\, W_3$. If the parameters $a_1 \in P_{\underline{\mathcal{C}}}(W_1)$, $a_{21}, a_{22} \in P_{\underline{\mathcal{C}}}(W_2)$ and $a_3 \in P_{\underline{\mathcal{C}}}(W_3)$ are such that $x \in FV(a_1)$, $\underline{x}, y \in FV(a_2)$ and $\underline{y} \in FV(a_3)$, then $U' = (W'_1\, {}_{x'}\square_{\underline{x}'}\, W'_2)\, {}_{y'}\square_{\underline{y}'}\, W'_3$, where
$$W_1[\underline{a_1^{\tau_1}}/\underline{a_1}] \equiv W'_1, \;\; W_2[\underline{a_{21}^{\tau_{21}}}/\underline{a_{21}}][\underline{a_{22}^{\tau_{22}}}/\underline{a_{22}}] \equiv W'_2 \;\text{ and }\; W_3[\underline{a_3^{\tau_3}}/\underline{a_3}] \equiv W'_3$$
and τ_1, τ_{21}, τ_{22} and τ_3 rename x to x', $\underline{x}$ to $\underline{x}'$, y to y' and $\underline{y}$ to $\underline{y}'$. We set
$$\varphi^{U'} = \beta^{x',\underline{x}';y',\underline{y}'}_{W'_1,W'_2,W'_3}.$$
We conclude by (EQ) and $(\beta\sigma)$, for $\sigma = id_X$.
- If $\varphi = {\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}}^{-1}$, then U' has the shape $W'_1\, {}_{x'}\square_{\underline{x}'}\, (W'_2\, {}_{y'}\square_{\underline{y}'}\, W'_3)$ (where W'_i and x', $\underline{x}'$, y' and $\underline{y}'$ are as in the previous case), and we set
$$\varphi^{U'} = {\beta^{x',\underline{x}';y',\underline{y}'}_{W'_1,W'_2,W'_3}}^{-1}.$$
We conclude by (EQ), $(\beta\sigma)$ and Remark 4.4.6(a), for $\sigma = id_X$.
- Suppose that $\varphi = \gamma^{x,y}_{W_1,W_2}$. The source of φ is then $U = W_1\, {}_x\square_y\, W_2$. If the parameters $a_1 \in P_{\underline{\mathcal{C}}}(W_1)$ and $a_2 \in P_{\underline{\mathcal{C}}}(W_2)$ are such that $x \in FV(a_1)$ and $y \in FV(a_2)$, then $U' = W'_1\, {}_{x'}\square_{y'}\, W'_2$, where,
$$W_1[\underline{a_1^{\tau_1}}/\underline{a_1}] \equiv W'_1 \;\text{ and }\; W_2[\underline{a_2^{\tau_2}}/\underline{a_2}] \equiv W'_2$$
and τ_1 and τ_2 rename x to x' and y to y', respectively. We set
$$\varphi^{U'} = \gamma^{x',y'}_{W'_1,W'_2}$$
and conclude by (EQ) and $(\gamma\sigma)$, for $\sigma = id_X$.
- Suppose that $\vdash \varphi_1 : U \to W$, $\vdash \varphi_2 : W' \to V$ and that $W \equiv W'$, and let $\varphi = \varphi_2 \circ \varphi_1$. By the induction hypothesis for φ_1 and U', there exist an arrow term
$$\varphi_1^{U'} : U' \to W_t(\varphi_1^{U'}),$$
such that $W_t(\varphi_1^{U'}) \equiv W$ and $[\varphi_1]_X = [\varphi_1^{U'}]_X$. Since $W \equiv W'$, by the transitivity of $\equiv$, we get $W_t(\varphi_1^{U'}) \equiv W'$. By the induction hypothesis for φ_2 and $W_t(\varphi_1^{U'})$, there exists an arrow term
$$\varphi_2^{W_t(\varphi_1^{U'})} : W_t(\varphi_1^{U'}) \to W_t(\varphi_2^{W_t(\varphi_1^{U'})}),$$

such that $W_t(\varphi_2^{W_t(\varphi_1^{U'})}) \equiv V$ and $[\varphi_2]_X = [\varphi_2^{W_t(\varphi_1^{U'})}]_X$. We define

$$\varphi^{U'} = \varphi_2^{W_t(\varphi_1^{U'})} \circ \varphi_1^{U'}.$$

- Suppose that $\vdash \varphi_1 : U_1 \to V_1$, $\vdash \varphi_2 : U_2 \to V_2$, and let $\varphi = \varphi_1 \,{}_x\square_y\, \varphi_2$. In this case, the source of φ is $U = U_1 \,{}_x\square_y\, U_2$ and we have two possibilities for the shape of U'.
  - $U' = U_1' \,{}_{x'}\square_{y'}\, U_2'$, where, assuming that $a_1 \in P_{\underline{\mathcal{C}}}(U_1)$ and $a_2 \in P_{\underline{\mathcal{C}}}(U_2)$ are such that $x \in FV(a_1)$ and $y \in FV(a_2)$, $U_1[\underline{a_1^{\tau_1}}/\underline{a_1}] \equiv U_1'$ and $U_2[\underline{a_2^{\tau_2}}/\underline{a_2}] \equiv U_2'$. Since $\underline{a_1^{\tau_1}} \in P_{\underline{\mathcal{C}}}(U_1')$ and $\underline{a_2^{\tau_2}} \in P_{\underline{\mathcal{C}}}(U_2')$, this means that, symmetrically, we have $U_1'[\underline{a_1}/\underline{a_1^{\tau_1}}] \equiv U_1$ and $U_2'[\underline{a_2}/\underline{a_2^{\tau_2}}] \equiv U_2$. By the induction hypothesis for φ_1 and $U_1'[\underline{a_1}/\underline{a_1^{\tau_1}}]$, as well as φ_2 and $U_2'[\underline{a_2}/\underline{a_2^{\tau_2}}]$, we get arrow terms

$$\varphi_1^{U_1'[\underline{a_1}/\underline{a_1^{\tau_1}}]} : U_1'[\underline{a_1}/\underline{a_1^{\tau_1}}] \to W_t(\varphi_1^{U_1'[\underline{a_1}/\underline{a_1^{\tau_1}}]})$$

    and

$$\varphi_2^{U_2'[\underline{a_2}/\underline{a_2^{\tau_2}}]} : U_2'[\underline{a_2}/\underline{a_2^{\tau_2}}] \to W_t(\varphi_2^{U_2'[\underline{a_2}/\underline{a_2^{\tau_2}}]}),$$

    such that

$$W_t(\varphi_1^{U_1'[\underline{a_1}/\underline{a_1^{\tau_1}}]}) \equiv V_1, \quad \text{and} \quad [\varphi_1]_X = [\varphi_1^{U_1'[\underline{a_1}/\underline{a_1^{\tau_1}}]}]_X$$

    and

$$W_t(\varphi_2^{U_2'[\underline{a_2}/\underline{a_2^{\tau_2}}]}) \equiv V_2 \quad \text{and} \quad [\varphi_2]_X = [\varphi_2^{U_2'[\underline{a_2}/\underline{a_2^{\tau_2}}]}]_X.$$

    By means of substitution on arrow terms, we define

$$\varphi^{U'} = \varphi_1^{U_1'[\underline{a_1}/\underline{a_1^{\tau_1}}]}[\underline{a_1^{\tau_1}}/\underline{a_1}] \,{}_{x'}\square_{y'}\, \varphi_2^{U_2'[\underline{a_2}/\underline{a_2^{\tau_2}}]}[\underline{a_2^{\tau_2}}/\underline{a_2}].$$

  - $U' = U_1' \,{}_x\square_y\, U_2'$, where $U_1 \equiv U_1'$ and $U_2 \equiv U_2'$. In this case, we define

$$\varphi^{U'} = \varphi_1^{U_1'} \,{}_x\square_y\, \varphi_2^{U_2'}.$$

We conclude by Lemma 4.16. ∎

The first reduction

We make the first reduction in two steps. We first define a (non-deterministic) rewriting algorithm $\rightsquigarrow$ on $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$ with outputs in $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$, in such a way that the interpretation of a term of $\mathtt{Free}_{\underline{\mathcal{C}}}$ matches the interpretations of (all) its normal forms relative to $\rightsquigarrow$. We then use Lemma 4.18 to move from $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$ to $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$, while preserving the equality of interpretations from the first step. This allows us to reduce the proof of the coherence theorem, which concerns all $\beta\gamma\sigma$-diagrams, to the consideration of parallel $\beta\gamma$-arrows in $\mathcal{C}(X)$ only.

We first define the rewriting algorithm $\rightsquigarrow$ on object terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$. The algorithm $\rightsquigarrow$ takes an object term $\mathcal{W}$ of $\mathtt{Free}_{\underline{\mathcal{C}}}$ and returns (non-deterministically) an object term W of $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$, which we denote by $\mathcal{W} \rightsquigarrow W$, in the way specified in Figure 4.4[1].

The formal system defined in Figure 4.4 is obviously terminating, in the sense that for all object terms $\mathcal{W}$ of $\mathtt{Free}_{\underline{\mathcal{C}}}$ there exists an object term W of $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$, such that $\mathcal{W} \rightsquigarrow W$. Notice also that the last rule is non-deterministic, as it involves a choice of x' and y'. In what follows,

[1]Strictly speaking, the formal system that we define is a term rewriting system on $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$, since the object terms of $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$ are clearly contained in $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$. We use the word *algorithm* to further emphasise that the normal forms (i.e. the outputs of the algorithm) themselves determine a *different syntax*, and, thereby, to stress the reduction process encoded by the formal system.

$$\frac{}{\underline{a} \rightsquigarrow \underline{a}} \qquad \frac{\mathcal{W}_1 \rightsquigarrow W_1 \qquad \mathcal{W}_2 \rightsquigarrow W_2}{\mathcal{W}_1 \,{}_x\square_y\, \mathcal{W}_2 \rightsquigarrow W_1 \,{}_x\square_y\, W_2}$$

$$\frac{}{\underline{a}^\sigma \rightsquigarrow \underline{a^\sigma}} \qquad \frac{\mathcal{W} \rightsquigarrow W}{\mathcal{W}^{id_X} \rightsquigarrow W} \qquad \frac{\mathcal{W}^{\sigma\circ\tau} \rightsquigarrow W}{(\mathcal{W}^\sigma)^\tau \rightsquigarrow W}$$

$$\frac{\begin{array}{ccc} \sigma : Z \to X\backslash\{x\}\cup Y\backslash\{y\} & x',y' \notin X\backslash\{x\}\cup Y\backslash\{y\} & x' \neq y' \\ \sigma_1 : \sigma^{-1}[X\backslash\{x\}]\cup\{x'\} \to X & \sigma_1|^{X\backslash\{x\}} = \sigma|^{X\backslash\{x\}} & \sigma_1(x') = x \\ \sigma_2 : \sigma^{-1}[Y\backslash\{y\}]\cup\{y'\} \to Y & \sigma_2|^{Y\backslash\{y\}} = \sigma|^{Y\backslash\{y\}} & \sigma_2(y') = y \\ \multicolumn{3}{c}{\mathcal{W}_1^{\sigma_1} \rightsquigarrow W_1 \quad \mathcal{W}_2^{\sigma_2} \rightsquigarrow W_2} \end{array}}{(\mathcal{W}_1 \,{}_x\square_y\, \mathcal{W}_2)^\sigma \rightsquigarrow W_1 \,{}_{x'}\square_{y'}\, W_2}$$

FIGURE 4.4: The rewriting algorithm $\rightsquigarrow$ on the object terms of $\texttt{Free}_{\underline{\mathcal{C}}}$

for an arbitrary object term $\mathcal{W}$ of $\texttt{Free}_{\underline{\mathcal{C}}}$, we shall say that the outputs of the algorithm $\rightsquigarrow$ applied on $\mathcal{W}$ are *normal forms of* $\mathcal{W}$ and we shall denote with $\texttt{NF}(\mathcal{W})$ the collection of all normal forms of $\mathcal{W}$.

The formal system $(\texttt{Free}_{\underline{\mathcal{C}}}, \rightsquigarrow)$ satisfies the following confluence-like property.

Lemma 4.19. *If $W_1, W_2 \in \texttt{NF}(\mathcal{W})$, then $W_1 \equiv W_2$.*

Proof. Suppose that $(\mathcal{W}_1 \,{}_x\square_y\, \mathcal{W}_2)^\sigma \rightsquigarrow W_1 \,{}_{x'}\square_{y'}\, W_2$ is obtained from $\mathcal{W}_1^{\sigma_1} \rightsquigarrow W_1$ and $\mathcal{W}_2^{\sigma_2} \rightsquigarrow W_2$, and $(\mathcal{W}_1 \,{}_x\square_y\, \mathcal{W}_2)^\sigma \rightsquigarrow W_1' \,{}_{x''}\square_{y''}\, W_2'$ from $\mathcal{W}_1^{\tau_1} \rightsquigarrow W_1'$ and $\mathcal{W}_2^{\tau_2} \rightsquigarrow W_2'$.

Let $a \in P_{\underline{\mathcal{C}}}(W_1)$ and $b \in P_{\underline{\mathcal{C}}}(W_2)$ be such that $FV(a) = X_1$, $FV(b) = Y_1$, $x \in X_1$ and $y \in Y_1$, and let $\kappa_1 : X_1\backslash\{x'\} \cup \{x''\} \to X_1$ be the renaming of x' to x'' and $\kappa_2 : Y_1\backslash\{y'\} \cup \{y''\} \to Y_1$ the renaming of y' to y''. It is then easy to show that $\mathcal{W}_1^{\tau_1} \rightsquigarrow W_1[\underline{a^{\kappa_1}}/\underline{a}]$ and $\mathcal{W}_2^{\tau_2} \rightsquigarrow W_2[\underline{a^{\kappa_2}}/\underline{a}]$.

By the definition of $\equiv$, and the induction hypothesis for $\mathcal{W}_1^{\tau_1}$ (that reduces to both W_1' and $W_1[\underline{a^{\kappa_1}}/\underline{a}]$) and $\mathcal{W}_2^{\tau_2}$ (that reduces to both W_2' and $W_2[\underline{b^{\kappa_2}}/\underline{b}]$), we then have

$$W_1 \,{}_{x'}\square_{y'}\, W_2 \equiv W_1[\underline{a^{\kappa_1}}/\underline{a}] \,{}_{x''}\square_{y''}\, W_2[\underline{b^{\kappa_2}}/\underline{b}] = W_1' \,{}_{x''}\square_{y''}\, W_2'.$$

■

Lemma 4.20. *For an arbitrary object term $\mathcal{W} : X$ of $\texttt{Free}_{\underline{\mathcal{C}}}$ and an arbitrary $W \in \texttt{NF}(\mathcal{W})$, the equality of interpretations $[[\mathcal{W}]]_X = [W]_X$ holds in $\mathcal{C}(X)$.*

Proof. By induction on the structure of $\mathcal{W}$.

- If $\mathcal{W} = \underline{a}$, we trivially have $[[\underline{a}]]_X = a = [\underline{a}]_X$.
- If $\mathcal{W} = \mathcal{W}_1{}_x\square_y\mathcal{W}_2$, where $\mathcal{W}_1 : X$ and $\mathcal{W}_2 : Y$, then, for any $W_1 \in \texttt{NF}(\mathcal{W}_1)$ and $W_2 \in \texttt{NF}(\mathcal{W}_2)$, $W_1 \,{}_x\square_y\, W_2 \in \texttt{NF}(\mathcal{W})$. Hence, by Lemma 4.19, we have that $W \equiv W_1 \,{}_x\square_y\, W_2$. By the induction hypothesis for $\mathcal{W}_1$ and $\mathcal{W}_2$, we have that $[[\mathcal{W}_1]]_X = [W_1]_X$ and $[[\mathcal{W}_2]]_Y = [W_2]_Y$, and, by Lemma 4.16, we get

$$\begin{aligned} [[\, \mathcal{W}_1{}_x\square_y\mathcal{W}_2 \,]]_{X\backslash\{x\}\cup Y\backslash\{y\}} &= [[\mathcal{W}_1]]_X \,{}_x\circ_y\, [[\mathcal{W}_2]]_Y \\ &= [W_1]_X \,{}_x\circ_y\, [W_2]_Y \\ &= [W_1 \,{}_x\square_y\, W_2]_{X\backslash\{x\}\cup Y\backslash\{y\}} \\ &= [W]_{X\backslash\{x\}\cup Y\backslash\{y\}}. \end{aligned}$$

- Suppose that $\mathcal{W} = \mathcal{V}^\sigma$, where $\mathcal{V} : X$ and $\sigma : Y \to X$. We proceed by case analysis relative to the shape of $\mathcal{V}$ (and σ).
  - If $\mathcal{V} = \underline{a}$, for some $a \in P_{\underline{\mathcal{C}}}$, then

$$[[\underline{a}^\sigma]]_Y = [[\underline{a}]]_X^\sigma = [\underline{a}]_X^\sigma = a^\sigma = [\underline{a^\sigma}]_Y.$$

– If $\sigma = id_X$, and if $V \in \mathtt{NF}(\mathcal{V})$, then $V \in \mathtt{NF}(\mathcal{W})$, and, by Lemma 4.19, we have that $W \equiv V$. By the induction hypothesis for $\mathcal{V}$ and Lemma 4.16, we get

$$[[\mathcal{V}^{id_X}]]_X = [[\mathcal{V}]]_X^{id_X} = [[\mathcal{V}]]_X = [V]_X = [W]_X.$$

– If $\mathcal{V} = \mathcal{V}_{1x}\square_y\mathcal{V}_2$, and if $V_1 \in \mathtt{NF}(\mathcal{V}_1^{\sigma_1})$ and $V_2 \in \mathtt{NF}(\mathcal{V}_2^{\sigma_2})$, then $V_{1x'}\square_{y'}V_2 \in \mathtt{NF}(\mathcal{W})$, and, by Lemma 4.19, $W \equiv V_{1x'}\square_{y'}V_2$. By the induction hypothesis for $\mathcal{V}_1^{\sigma_1}$ and $\mathcal{V}_2^{\sigma_2}$ and Lemma 4.16, we get

$$\begin{aligned}
[[(\mathcal{V}_1\,{}_x\square_y\,\mathcal{V}_2)^{\sigma}]]_Y &= [[\mathcal{V}_1\,{}_x\square_y\,\mathcal{V}_2]]_X^{\sigma}\\
&= ([[\mathcal{V}_1]]_{X_1}\,{}_x\square_y\,[[\mathcal{V}_2]]_{X_2})^{\sigma}\\
&= [[\mathcal{V}_1]]_{X_1}^{\sigma_1}\,{}_{x'}\square_{y'}\,[[\mathcal{V}_2]]_{X_2}^{\sigma_2}\\
&= [[\mathcal{V}_1^{\sigma_1}]]_{Y_1}\,{}_{x'}\square_{y'}\,[[\mathcal{V}_2^{\sigma_2}]]_{Y_2}\\
&= [V_1]_{Y_1}\,{}_{x'}\square_{y'}\,[V_2]_{Y_2}\\
&= [V_1\,{}_{x'}\square_{y'}\,V_2]_Y\\
&= [W]_Y
\end{aligned}$$

– If $\mathcal{V} = \mathcal{U}^{\tau}$, and if $U \in \mathtt{NF}(\mathcal{U}^{\tau\circ\sigma})$, then $U \in \mathtt{NF}(\mathcal{W})$, and, by Lemma 4.19, $W \equiv U$. By the induction hypothesis for $\mathcal{U}^{\tau\circ\sigma}$ and Lemma 4.16, we get

$$[[(\mathcal{U}^{\tau})^{\sigma}]]_Y = ([[\mathcal{U}]]_X^{\tau})^{\sigma} = [[\mathcal{U}]]_X^{\tau\circ\sigma} = [[\mathcal{U}^{\tau\circ\sigma}]]_Y = [U]_Y = [W]_Y. \qquad \blacksquare$$

We move on to the first step of the first reduction of arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$: in Figure 4.5, we define a (non-deterministic) rewriting algorithm $\rightsquigarrow$, which "normalises" arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$.

We make first observations about this rewriting algorithm.

Remark 4.21. *Notice that, if $\Phi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ and if $\Phi \rightsquigarrow \varphi$, then $\vdash \varphi : W_1 \rightarrow W_2$, for some $W_1 \in \mathtt{NF}(\mathcal{W}_1)$ and $W_2 \in \mathtt{NF}(\mathcal{W}_2)$. Also, in the rule defining $(\Phi_2 \circ \Phi_1)^{\sigma} \rightsquigarrow \varphi_2 \circ \varphi_1$, the arrow term $\varphi_2 \circ \varphi_1$ is not well-typed in $\underline{\mathtt{Free}_{\mathcal{C}}}$ in general.*

As it was the case for the algorithm on object terms, this formal system is terminating. Therefore, the algorithm gives us, for each arrow term $\Phi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$, the set $\mathtt{NF}(\Phi)$ of normal forms of Φ, which are arrow terms of $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}$. Here is the most important property of these normal forms.

Lemma 4.22. *For arbitrary arrow term Φ of $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$ and $\varphi \in \mathtt{NF}(\Phi)$, the equality of interpretations $[[\Phi]]_X = [\varphi]_X$ holds in $\mathcal{C}(X)$.*

Proof. By induction on the structure of Φ and Lemma 4.20. $\blacksquare$

Suppose that, for all object terms $\mathcal{W}$ of $\mathtt{Free}_{\underline{\mathcal{C}}}$, a normal form $\mathtt{red}_1(\mathcal{W}) \in \mathtt{NF}(\mathcal{W})$ in $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}$ has been fixed, and that, *independently of that choice*, for all arrow terms Φ of $\mathtt{Free}_{\underline{\mathcal{C}}}$ a normal form $\mathtt{red}_1(\Phi) \in \mathtt{NF}(\Phi)$ in $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}$ has been fixed.

We define the *first reduction function* $\mathtt{Red}_1 : \mathtt{Free}_{\underline{\mathcal{C}}} \rightarrow \underline{\mathtt{Free}_{\mathcal{C}}}$ by

$$\mathtt{Red}_1(\mathcal{W}) = \mathtt{red}_1(\mathcal{W}) \quad \text{and} \quad \mathtt{Red}_1(\Phi) = \mathtt{red}_1(\Phi)^{\mathtt{red}_1(\mathcal{W}_1)},$$

where $\Phi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$. Observe that, in the definition of $\mathtt{Red}_1(\Phi)$, we used the construction of Lemma 4.18, which indeed turns $\mathtt{red}_1(\Phi)$ (which is an arrow term of $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}$) into an arrow term of $\underline{\mathtt{Free}_{\mathcal{C}}}$. Also, for an arrow term $\Phi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ of $\mathtt{Free}_{\underline{\mathcal{C}}}$, we have that $\mathtt{Red}_1(\Phi) : \mathtt{Red}_1(\mathcal{W}_1) \rightarrow W_2$, where, in general, $W_2 \neq \mathtt{Red}_1(\mathcal{W}_2)$. However, the following important property holds.

$$\frac{U \in \mathtt{NF}(\mathcal{U})}{1_{\mathcal{U}} \rightsquigarrow 1_U}$$

$$\frac{W_i \in \mathtt{NF}(\mathcal{W}_i) \quad i \in \{1,2,3\}}{\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \rightsquigarrow \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}} \qquad \frac{W_i \in \mathtt{NF}(\mathcal{W}_i) \quad i \in \{1,2,3\}}{\beta^{x,\underline{x};y,\underline{y}\;\;-1}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \rightsquigarrow \beta^{x,\underline{x};y,\underline{y}\;\;-1}_{W_1,W_2,W_3}}$$

$$\frac{W_i \in \mathtt{NF}(\mathcal{W}_i) \quad i \in \{1,2\}}{\gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2} \rightsquigarrow \gamma^{x,y}_{W_1,W_2}}$$

$$\frac{}{\varepsilon 1^{\sigma}_{\underline{a}} \rightsquigarrow 1_{\underline{a^{\sigma}}}} \qquad \frac{}{\varepsilon 1^{\sigma\;-1}_{\underline{a}} \rightsquigarrow 1_{\underline{a^{\sigma}}}}$$

$$\frac{W \in \mathtt{NF}(\mathcal{W})}{\varepsilon 2_{\mathcal{W}} \rightsquigarrow 1_W} \qquad \frac{W \in \mathtt{NF}(\mathcal{W})}{\varepsilon 2^{-1}_{\mathcal{W}} \rightsquigarrow 1_W}$$

$$\frac{W \in \mathtt{NF}(\mathcal{W}^{\sigma\circ\tau})}{\varepsilon 3^{\sigma,\tau}_{\mathcal{W}} \rightsquigarrow 1_W} \qquad \frac{W \in \mathtt{NF}(\mathcal{W}^{\sigma\circ\tau})}{\varepsilon 3^{\sigma,\tau\;-1}_{\mathcal{W}} \rightsquigarrow 1_W}$$

$$\frac{W \in \mathtt{NF}((\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^{\sigma})}{\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} \rightsquigarrow 1_W} \qquad \frac{W_1 \in \mathtt{NF}(\mathcal{W}_1^{\sigma_1}) \quad W_2 \in \mathtt{NF}(\mathcal{W}_2^{\sigma_2})}{\varepsilon 4^{x,y;x',y'\;-1}_{\mathcal{W}_1,\mathcal{W}_2;\sigma_1,\sigma_2} \rightsquigarrow 1_W}$$

$$\frac{\Phi_1 \rightsquigarrow \varphi_1 \quad \Phi_2 \rightsquigarrow \varphi_2}{\Phi_2 \circ \Phi_1 \rightsquigarrow \varphi_2 \circ \varphi_1} \qquad \frac{\Phi_1 \rightsquigarrow \varphi_1 \quad \Phi_2 \rightsquigarrow \varphi_2}{\Phi_1\;{}_x\square_y\;\Phi_2 \rightsquigarrow \varphi_1\;{}_x\square_y\;\varphi_2}$$

$$\frac{}{1^{\sigma}_{\underline{a}} \rightsquigarrow 1_{\underline{a^{\sigma}}}}$$

$$\frac{W_i \in \mathtt{NF}(\mathcal{W}_i^{\sigma_i})}{(\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3})^{\sigma} \rightsquigarrow \beta^{x',\underline{x}';y',\underline{y}'}_{W_1,W_2,W_3}} \qquad \frac{W_i \in \mathtt{NF}(\mathcal{W}_i^{\sigma_i})}{(\beta^{x,\underline{x};y,\underline{y}\;\;-1}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3})^{\sigma} \rightsquigarrow \beta^{x',\underline{x}';y',\underline{y}'\;\;-1}_{W_1,W_2,W_3}}$$

$$\frac{W_i \in \mathtt{NF}(\mathcal{W}_i^{\sigma_i}) \quad i \in \{1,2\}}{(\gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2})^{\sigma} \rightsquigarrow \gamma^{x',y'}_{W_1,W_2}}$$

$$\frac{}{(\varepsilon 1^{\sigma}_{\underline{a}})^{\kappa} \rightsquigarrow 1_{\underline{a^{\sigma\circ\kappa}}}} \qquad \frac{}{(\varepsilon 1^{\sigma\;-1}_{\underline{a}})^{\kappa} \rightsquigarrow 1_{\underline{a^{\sigma\circ\kappa}}}}$$

$$\frac{W \in \mathtt{NF}(\mathcal{W}^{\kappa})}{(\varepsilon 2_{\mathcal{W}})^{\kappa} \rightsquigarrow 1_W} \qquad \frac{W \in \mathtt{NF}(\mathcal{W}^{\kappa})}{(\varepsilon 2^{-1}_{\mathcal{W}})^{\kappa} \rightsquigarrow 1_W}$$

$$\frac{W \in \mathtt{NF}(\mathcal{W}^{\sigma\circ\tau\circ\kappa})}{(\varepsilon 3^{\sigma,\tau}_{\mathcal{W}})^{\kappa} \rightsquigarrow 1_W} \qquad \frac{W \in \mathtt{NF}(\mathcal{W}^{\sigma\circ\tau\circ\kappa})}{(\varepsilon 3^{\sigma,\tau\;-1}_{\mathcal{W}})^{\kappa} \rightsquigarrow 1_W}$$

$$\frac{W \in \mathtt{NF}((\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^{\sigma\circ\kappa})}{(\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma})^{\kappa} \rightsquigarrow 1_W} \qquad \frac{W_1 \in \mathtt{NF}(\mathcal{W}_1^{\sigma_1\circ\kappa_1}) \quad W_2 \in \mathtt{NF}(\mathcal{W}_2^{\sigma_2\circ\kappa_2})}{(\varepsilon 4^{x,y;x',y'\;-1}_{\mathcal{W}_1,\mathcal{W}_2;\sigma})^{\kappa} \rightsquigarrow 1_W}$$

$$\frac{\Phi \rightsquigarrow \varphi}{\Phi^{id_X} \rightsquigarrow \varphi} \qquad \frac{\Phi^{\sigma\circ\tau} \rightsquigarrow \varphi}{(\Phi^{\sigma})^{\tau} \rightsquigarrow \varphi} \qquad \frac{\Phi_1^{\sigma_1} \rightsquigarrow \varphi_1 \quad \Phi_2^{\sigma_2} \rightsquigarrow \varphi_2}{(\Phi_1\;{}_x\square_y\;\Phi_2)^{\sigma} \rightsquigarrow \varphi_1\;{}_{x'}\square_{y'}\;\varphi_2} \qquad \frac{\Phi_1^{\sigma} \rightsquigarrow \varphi_1 \quad \Phi_2^{\sigma} \rightsquigarrow \varphi_2}{(\Phi_2 \circ \Phi_1)^{\sigma} \rightsquigarrow \varphi_2 \circ \varphi_1}$$

FIGURE 4.5: The rewriting algorithm $\rightsquigarrow$ on the arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$

Lemma 4.23. *For any two arrow terms $\Phi, \Psi : \mathcal{W}_1 \to \mathcal{W}_2$ of the same type in $\mathtt{Free}_{\mathcal{C}}$, $\mathtt{Red}_1(\Phi)$ and $\mathtt{Red}_1(\Psi)$ are arrow terms of the same type in $\underline{\mathtt{Free}_{\mathcal{C}}}$.*

Proof. That $\mathtt{Red}_1(\Phi)$ and $\mathtt{Red}_1(\Psi)$ have the same source is clear by the definition. We prove the equality $W_t(\mathtt{Red}_1(\Phi)) = W_t(\mathtt{Red}_1(\Psi))$ by induction on the proof of $W_t(\mathtt{Red}_1(\Phi)) \equiv W_t(\mathtt{Red}_1(\Psi))$.

Suppose that

$$W_t(\mathtt{Red}_1(\Phi)) = W_1 \,{}_x\square_y\, W_2 \quad \text{and} \quad W_t(\mathtt{Red}_1(\Psi)) = W_1[\underline{a^{\tau_1}}/\underline{a}] \,{}_{x'}\square_{y'}\, W_2[\underline{b^{\tau_2}}/\underline{b}].$$

If, moreover, at least one of τ_1 and τ_2 is not the identitiy, i.e. if, say, $x' \neq x$, then, by Remark 4.11, it cannot be the case that $\mathtt{Red}_1(\Phi)$ and $\mathtt{Red}_1(\Psi)$ have the same source, which is a contradiction. ∎

The following theorem, essential for the proof of the coherence theorem, is simply an instance of Lemma 4.20 and Lemma 4.22.

Theorem 4.24. *For an arbitrary object term $\mathcal{W}$ and an arbitrary arrow term Φ of $\mathtt{Free}_{\underline{\mathcal{C}}}$, the equalities of interpretations*

$$[[\mathcal{W}]]_X = [\mathtt{Red}_1(\mathcal{W})]_X \quad \textit{and} \quad [[\Phi]]_X = [\mathtt{Red}_1(\Phi)]_X$$

hold in $\mathcal{C}(X)$.

4.1.4 The second reduction: getting rid of the cyclicity

Intuitively, this reduction goes from "cyclic operadic" to just "operadic", which cuts down the problem of commutation of all $\beta\gamma$-diagrams of $\mathcal{C}(X)$ to the problem of commutation of all $\beta\vartheta$-diagrams of $\mathcal{C}(X)$ (see (4.1.1)). As the "removal of cyclicity" is based on a transition from unrooted to rooted trees, we shall use a tree representation of our syntax, more convenient for "visualising" this reduction. The latter representation builds on the formalism of unrooted trees introduced in §2.3.1. Given that $\mathcal{C}$ is a non-unital categorified cyclic operad, in what follows we shall consider only *ordinary unrooted trees*, to which we shall refer simply as unrooted trees. Moreover, since the purpose of the formalism is to provide a representation of the terms of $\underline{\mathtt{Free}_{\mathcal{C}}}$, which do not encode symmetries, the unrooted trees will *not* be quotiented with α-equivalence. In this regard, we shall denote with $\mathtt{T}_{\underline{\mathcal{C}}}$ (resp. $\mathtt{T}_{\underline{\mathcal{C}}}(X)$) the collection of all unrooted trees whose corollas belong to $P_{\underline{\mathcal{C}}}$ (resp. whose corollas belong to $P_{\underline{\mathcal{C}}}$ and whose free variables are given by the set X).

A tree-wise representation of the terms of $\underline{\mathtt{Free}_{\mathcal{C}}}$: the syntax $\mathtt{T}^{+}_{\underline{\mathcal{C}}}$

We first introduce the syntax of parenthesised words generated by $P_{\underline{\mathcal{C}}}$, as

$$\boxed{w ::= \underline{a} \,|\, ww}$$

where $a \in P_{\underline{\mathcal{C}}}$. We shall denote the collection of all terms obtained in this way by $\mathtt{PWords}_{\underline{\mathcal{C}}}$.

For an unrooted tree $\mathcal{T}$, we next introduce the *$\mathcal{T}$-admissibility* relation on $\mathtt{PWords}_{\underline{\mathcal{C}}}$. Recall from §2.3.1 that we write $\mathcal{T} = \{\mathcal{T}_1\,(xy)\,\mathcal{T}_2\}$ to denote that $\mathcal{T}_1$ and $\mathcal{T}_2$ make a decompostion of $\mathcal{T}$ along the edge (xy). Intuitively, w is $\mathcal{T}$-admissible if it represents a gradual composition of the corollas of $\mathcal{T}$. Formally, the predicate *w is $\mathcal{T}$-admissible* is defined by the following two clauses:

- ◇ $\underline{a}$ is $\mathcal{T}$-admissible if $Cor(\mathcal{T}) = \{a\}$, and
- ◇ if $\mathcal{T} = \{\mathcal{T}_1\,(xy)\,\mathcal{T}_2\}$, w_1 is $\mathcal{T}_1$-admissible and w_2 is $\mathcal{T}_2$-admissible, then w_1w_2 is $\mathcal{T}$-admissible.

We shall denote the set of all $\mathcal{T}$-admissible terms of $\mathtt{PWords}_{\underline{\mathcal{C}}}$ with $A(\mathcal{T})$.

Remark 4.25. *Notice that, if w is $\mathcal{T}$-admissible, then, since all the corollas of $\mathcal{T}$ are mutually distinct, w does not contain repetitions of letters from $P_{\underline{\mathcal{C}}}$.*

A parenthesised word can be admissible with respect to more than one unrooted tree. Concretely, in the second clause above, w_1w_2 is admissible with respect to any *unrooted tree whose decomposition is $\{\mathcal{T}_1, \mathcal{T}_2\}$.*

We introduce the syntax of *unrooted trees with grafting data induced by* $\underline{\mathcal{C}}$, denoted by $\mathtt{T}^{+}_{\underline{\mathcal{C}}}$, as follows. The collection of object terms of $\mathtt{T}^{+}_{\underline{\mathcal{C}}}$ is obtained by combining the syntax $\mathtt{T}_{\underline{\mathcal{C}}}$ and the syntax $\mathtt{PWords}_{\underline{\mathcal{C}}}$, by means of the $\mathcal{T}$-admissibility relation: we take for object terms of $\mathtt{T}^{+}_{\underline{\mathcal{C}}}$ all the pairs $(\mathcal{T}, w)$, typed as

$$\boxed{\dfrac{\mathcal{T} \in \mathtt{T}_{\underline{\mathcal{C}}}(X) \quad w \in \mathtt{PWords}_{\underline{\mathcal{C}}} \quad w \in A(\mathcal{T})}{(\mathcal{T}, w) : X}}$$

The arrow terms of $\mathtt{T}^{+}_{\underline{\mathcal{C}}}$ are obtained from raw terms

$$\boxed{\varphi ::= 1_{(\mathcal{T},w)} \mid \beta^{x,\underline{x};y,\underline{y}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)} \mid \beta^{x,\underline{x};y,\underline{y}\ \ -1}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)} \mid \gamma^{x,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)} \mid \varphi \circ \varphi \mid \varphi \;{}_{x}\square_{y}\; \varphi}$$

by typing them as shown in Figure 4.6.

$$\dfrac{}{1_{(\mathcal{T},w)} : (\mathcal{T}, w) \to (\mathcal{T}, w)}$$

$$\dfrac{\mathcal{T} = \{\{\mathcal{T}_1\,(x\underline{x})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y \in FV(\mathcal{T}_2)}{\beta^{x,\underline{x};y,\underline{y}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)} : (\mathcal{T}, (w_1w_2)w_3\,) \to (\mathcal{T}, w_1(w_2w_3)\,)}$$

$$\dfrac{\mathcal{T} = \{\mathcal{T}_1\,(x\underline{x})\,\{\mathcal{T}_2\,(y\underline{y})\,\mathcal{T}_3\}\} \quad \underline{x} \in FV(\mathcal{T}_2)}{\beta^{x,\underline{x};y,\underline{y}\ \ -1}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)} : (\mathcal{T}, w_1(w_2w_3)\,) \to (\mathcal{T}, (w_1w_2)w_3\,)}$$

$$\dfrac{\mathcal{T} = \{\mathcal{T}_1\,(x\underline{x})\,\mathcal{T}_2\}}{\gamma^{x,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)} : (\mathcal{T}, w_1w_2\,) \to (\mathcal{T}, w_2w_1\,)}$$

$$\dfrac{\varphi_1 : (\mathcal{T}, w_1) \to (\mathcal{T}, w_2) \quad \varphi_2 : (\mathcal{T}, w_2) \to (\mathcal{T}, w_3)}{\varphi_2 \circ \varphi_1 : (\mathcal{T}, w_1) \to (\mathcal{T}, w_3)}$$

$$\dfrac{\varphi_1 : (\mathcal{T}_1, w_1) \to (\mathcal{T}_1, w_1') \quad \varphi_2 : (\mathcal{T}_2, w_2) \to (\mathcal{T}_2, w_2')}{\varphi_1 \;{}_{x}\square_{y}\; \varphi_2 : (\{\mathcal{T}_1\,(xy)\,\mathcal{T}_2\}, w_1w_2) \to (\{\mathcal{T}_1\,(xy)\,\mathcal{T}_2\}, w_1'w_2')}$$

FIGURE 4.6: Typing rules for the arrow terms of $\mathtt{T}^{+}_{\underline{\mathcal{C}}}$

We shall denote the class of object terms of $\mathtt{T}^{+}_{\underline{\mathcal{C}}}$ whose type is X, together with the class of arrow terms whose types are pairs of object terms of type X, by $\mathtt{T}^{+}_{\underline{\mathcal{C}}}(X)$.

Lemma 4.26. *The terms of* $\mathtt{T}^{+}_{\underline{\mathcal{C}}}(X)$ *are in one-to-one corresponcence with the terms of* $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$.

Proof. The correspondence $\Delta_X : \mathtt{T}^{+}_{\underline{\mathcal{C}}}(X) \to \underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$ is defined recursively as follows:

- $\Delta_X((\{a(x_1,\dots,x_n); id_X\}, \underline{a})) = \underline{a}$,
- if $\Delta_X((\mathcal{T}_1, w_1)) = W_1$ and $\Delta_Y((\mathcal{T}_2, w_2)) = W_2$, and if $\mathcal{T} = \{\mathcal{T}_1(xy)\mathcal{T}_2\}$, then
$$\Delta_{X\backslash\{x\}\cup Y\backslash\{y\}}((\mathcal{T}, w_1w_2)) = W_1 \;{}_{x}\square_{y}\; W_2,$$
- $\Delta_X(1_{(\mathcal{T},w)}) = 1_{\Delta_X((\mathcal{T},w))}$,
- $\Delta_X(\beta^{x,\underline{x};y,\underline{y}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}) = \beta^{x,\underline{x};y,\underline{y}}_{\Delta_{X_1}((\mathcal{T}_1,w_1)),\Delta_{X_2}((\mathcal{T}_2,w_2)),\Delta_{X_2}((\mathcal{T}_3,w_3))}$,
- $\Delta_X(\beta^{x,\underline{x};y,\underline{y}\ \ -1}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}) = \beta^{x,\underline{x};y,\underline{y}\ \ -1}_{\Delta_{X_1}((\mathcal{T}_1,w_1)),\Delta_{X_2}((\mathcal{T}_2,w_2)),\Delta_{X_2}((\mathcal{T}_3,w_3))}$,

- $\Delta_X(\gamma^{x,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}) = \gamma^{x,y}_{\Delta_{X_1}((\mathcal{T}_1,w_1)),\Delta_{X_2}((\mathcal{T}_2,w_2))}$,
- $\Delta_X(\varphi_2 \circ \varphi_1) = \Delta_X(\varphi_2) \circ \Delta_X(\varphi_1)$,
- $\Delta_X(\varphi_1 \,{}_x\square_y\, \varphi_2) = \Delta_{X_1}(\varphi_1) \,{}_x\square_y\, \Delta_{X_2}(\varphi_2)$. ■

We define the interpretation function

$$\lfloor - \rfloor_X : \underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$$

to be the composition $[-]_X \circ \Delta_X$. The following lemma is an immediate consequence of the definition of $\lfloor - \rfloor_X$.

Lemma 4.27. *For arbitrary object term W and arrow term φ of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$, the equalities of interpretations*

$$[W]_X = \lfloor \Delta_X^{-1}(W) \rfloor_X \quad \textit{and} \quad [\varphi]_X = \lfloor \Delta_X^{-1}(\varphi) \rfloor_X$$

hold in $\mathcal{C}(X)$.

Lemma 4.26 and Lemma 4.27 justify the representation of terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ by means of unrooted trees with grafting data.

"Rooting" the syntax $\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}$: the syntax $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}$

In this part, we introduce the syntax of *rooted trees with grafting data induced by $\underline{\mathcal{C}}$*, denoted by $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}$, as follows.

For a pair $(\mathcal{T}, x)$ of an unrooted tree $\mathcal{T} \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$ and $x \in X$, we first introduce the $(\mathcal{T}, x)$-*admissibility* relation on $\mathtt{PWords}_{\underline{\mathcal{C}}}$. The predicate *$w$ is $(\mathcal{T}, x)$-admissibile* is defined by the following two clauses:

- $\underline{a}$ is $(\mathcal{T}, x)$-admissibile if $Cor(\mathcal{T}) = \{a\}$, and
- if $\mathcal{T} = \{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\}$, $x \in FV(\mathcal{T}_1)$ (without loss of generality), w_1 is $(\mathcal{T}_1, x)$-admissible and w_2 is $(\mathcal{T}_2, y)$-admissible, then $w_1 w_2$ is $(\mathcal{T}, x)$-admissible.

We shall denote the set of all $(\mathcal{T}, x)$-admissible terms of $\mathtt{PWords}_{\underline{\mathcal{C}}}$ with $A(\mathcal{T}, x)$.

Intuitively, w is $(\mathcal{T}, x)$-admissible if it is $\mathcal{T}$-admissible and it is an *operadic word* with respect to the rooted tree determined by considering x as the root of $\mathcal{T}$, in the sense that $(\mathcal{T}, w)$ enjoys the following normalisation property, inherent to (formal terms which describe) *operadic* operations: all β^{-1}-reduction sequences starting from $(\mathcal{T}, w)$ end with an object term $(\mathcal{T}, w')$, such that all pairs of parentheses of w' are associated to the left.

The object terms of $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}$ are triplets $(\mathcal{T}, x, w)$, typed as

$$\boxed{\dfrac{\mathcal{T} \in \underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X) \qquad x \in X \qquad w \in A(\mathcal{T}, x)}{(\mathcal{T}, x, w) : X}}$$

The class of arrow terms of $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}$ is obtained from raw terms

$$\boxed{\chi ::= \begin{cases} 1_{(\mathcal{T},x,w)} \mid \beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} \mid \beta^{z;y\ \ -1}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} \\ \theta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} \mid \chi \circ \chi \mid \chi \,{}_x\square_y\, \chi \end{cases}}$$

by typing them as shown in Figure 4.7.

$$\frac{}{1_{(\mathcal{T},x,w)} : (\mathcal{T},x,w) \to (\mathcal{T},x,w)}$$

$$\frac{\mathcal{T} = \{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y \in FV(\mathcal{T}_2) \quad x \in X \cap FV(\mathcal{T}_1)}{\beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} : (\mathcal{T},x,(w_1w_2)w_3\,) \to (\mathcal{T},x,w_1(w_2w_3)\,)}$$

$$\frac{\mathcal{T} = \{\mathcal{T}_1\,(z\underline{z})\,\{\mathcal{T}_2\,(y\underline{y})\,\mathcal{T}_3\}\} \quad \underline{z} \in FV(\mathcal{T}_2) \quad x \in X \cap FV(\mathcal{T}_1)}{\beta^{z;y\ \ -1}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} : (\mathcal{T},x,w_1(w_2w_3)\,) \to (\mathcal{T},x,(w_1w_2)w_3\,)}$$

$$\frac{\mathcal{T} = \{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y \in FV(\mathcal{T}_1) \quad x \in X \cap FV(\mathcal{T}_1)}{\theta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} : (\mathcal{T},x,(w_1w_2)w_3\,) \to (\mathcal{T},x,(w_1w_3)w_2\,)}$$

$$\frac{\chi_1 : (\mathcal{T},x,w_1) \to (\mathcal{T},x,w_2) \quad \chi_2 : (\mathcal{T},x,w_2) \to (\mathcal{T},x,w_3)}{\chi_2 \circ \chi_1 : (\mathcal{T},x,w_1) \to (\mathcal{T},x,w_3)}$$

$$\frac{\chi_1 : (\mathcal{T}_1,x,w_1) \to (\mathcal{T}_1,x,w'_1) \ \ \chi_2 : (\mathcal{T}_2,y,w_2) \to (\mathcal{T}_2,y,w'_2) \ \ z \in FV(\mathcal{T}_1) \ \ z \neq x}{\chi_1\, {}_z\square_y\, \chi_2 : (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x,w_1w_2) \to (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x,w'_1w'_2)}$$

FIGURE 4.7: Typing rules for the arrow terms of $\mathtt{rT}^{+}_{\underline{\mathcal{C}}}$

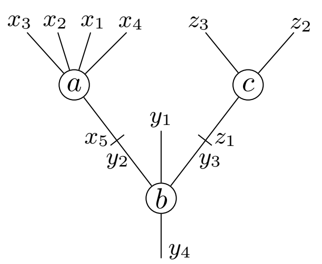
We shall denote the class of object terms of $\mathtt{rT}^{+}_{\underline{\mathcal{C}}}$ whose type is X, together with the class of arrow terms whose types are pairs of object terms of type X, by $\mathtt{rT}^{+}_{\underline{\mathcal{C}}}(X)$.

Notice that, for an object term $(\mathcal{T},x,w)$ of $\mathtt{rT}^{+}_{\underline{\mathcal{C}}}(X)$, the choice of $x \in X$ as the root of $\mathcal{T}$ determines the roots of all subtrees of $\mathcal{T}$, and, in particular, of all corollas of $\mathcal{T}$. In other words, this choice allows us to speak about the inputs and the output of any subtree of $\mathcal{T}$.

Formally, for a subtree $\mathcal{S}$ of $\mathcal{T}$ and a variable $x \in FV(\mathcal{T})$, we define the *set* $\mathtt{inp}_{(\mathcal{T},x)}(\mathcal{S})$ *of inputs* of $\mathcal{S}$ and the *output* $\mathtt{out}_{(\mathcal{T},x)}(\mathcal{S})$ *of* $\mathcal{S}$, induced by x, as follows. Let $a \in Cor(\mathcal{T})$ be such that $x \in FV(a)$. Then,

- ◇ if $a \in Cor(\mathcal{S})$, we have $\mathtt{inp}_{(\mathcal{T},x)}(\mathcal{S}) = FV(\mathcal{S})\backslash\{x\}$ and $\mathtt{out}_{(\mathcal{T},x)}(\mathcal{S}) = x$,
- ◇ if $a \notin Cor(\mathcal{S})$ and $c \in Cor(\mathcal{S})$ is the corolla of $\mathcal{S}$ whose path to a (see Remark 2.12) has the smallest length (among all paths from the corollas of $\mathcal{S}$ to a), then $\mathtt{inp}_{(\mathcal{T},x)}(\mathcal{S}) = FV(\mathcal{S})\backslash\{z\}$, where $z \in FV(c) \cap p$, and $\mathtt{out}_{(\mathcal{T},x)}(\mathcal{S}) = z$.

EXAMPLE 4.28. For the unrooted tree $\mathcal{T}$ from EXAMPLE 2.10, the choice of, say, $y_4 \in X$, turns $\mathcal{T}$ into a rooted tree, which can be visualised as



We have $\mathtt{inp}_{(\mathcal{T},y_4)}(b) = \{y_1,y_2,y_3\}$, $\mathtt{out}_{(\mathcal{T},y_4)}(b) = y_4$, $\mathtt{inp}_{(\mathcal{T},y_4)}(a) = \{x_1,x_2,x_3,x_4\}$, $\mathtt{out}_{(\mathcal{T},y_4)}(a) = x_5$ and $\mathtt{inp}_{(\mathcal{T},y_4)}(c) = \{z_2,z_3\}$, $\mathtt{out}_{(\mathcal{T},y_4)}(c) = z_1$.

Observe that, among all parenthesised words admissible with respect to $\mathcal{T}$, only $(\underline{ba})\underline{c}$ and $(\underline{bc})\underline{a}$ are operadic, relative to the choice of y_4 as the root of $\mathcal{T}$. □

The interpretation of $\mathtt{rT}^{+}_{\underline{\mathcal{C}}}$ in $\mathcal{C}$

The interpretation function

$$\lceil - \rceil_X : \mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$$

is defined recursively as follows:

- $\lceil(\{a(x_1,\dots,x_n)\}, x_i, \underline{a})\rceil_X = a$,
- $\lceil(\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\}, x, w_1w_2)\rceil_X = \lceil(\mathcal{T}_1, x, w_1)\rceil_{X_1}\,{}_z\circ_y \lceil(\mathcal{T}_2, y, w_2)\rceil_{X_2}$,

and

- $\lceil 1_{(\mathcal{T},x,w)}\rceil_X = 1_{\lceil(\mathcal{T},x,w)\rceil_X}$,
- $\lceil \beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}\rceil_X = \beta^{z,\underline{z};y,\underline{y}}_{\lceil(\mathcal{T}_1,x,w_1)\rceil_{X_1},\lceil(\mathcal{T}_2,\underline{z},w_2)\rceil_{X_2},\lceil(\mathcal{T}_3,\underline{z},w_3)\rceil_{X_3}}$,
- $\lceil \beta^{z;y\ -1}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}\rceil_X = \beta^{z,\underline{z};y,\underline{y}\ -1}_{\lceil(\mathcal{T}_1,x,w_1)\rceil_{X_1},\lceil(\mathcal{T}_2,\underline{z},w_2)\rceil_{X_2},\lceil(\mathcal{T}_3,\underline{z},w_3)\rceil_{X_3}}$,
- $\lceil \theta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}\rceil_X = \vartheta^{z,\underline{z};y,\underline{y}}_{\lceil(\mathcal{T}_1,x,w_1)\rceil_{X_1},\lceil(\mathcal{T}_2,\underline{z},w_2)\rceil_{X_2},\lceil(\mathcal{T}_3,\underline{y},w_3)\rceil_{X_3}}$ (see (4.1.1)),
- $\lceil \chi_2 \circ \chi_1\rceil_X = \lfloor \chi_2\rfloor_X \circ \lfloor \chi_1\rfloor_X$, and
- $\lceil \chi_1\,{}_z\square_y\,\chi_2\rceil_X = \lceil\chi_1\rceil_{X_1}\,{}_z\circ_y\lceil\chi_2\rceil_{X_2}$.

Remark 4.29. *Notice that $\lceil\chi\rceil_X$ is an arrow in $\mathcal{C}(X)$ all of whose instances of the isomorphism γ get "hidden" by using explicitly the abbreviation ϑ. In other words, the semantics of arrow terms of $\mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$ is what distinguishes $\beta\vartheta$-arrows of $\mathcal{C}(X)$.*

The second reduction

We define the familly of *second reduction functions*

$$\mathtt{Red}_2(X,x) : \underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X) \to \mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X),$$

where $x \in X$, as follows.

For the object terms of $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X)$, we set

$$\mathtt{Red}_2(X,x)((\mathcal{T},w)) = (\mathcal{T}, x, w^{\cdot x}),$$

where $w^{\cdot x}$ is the $(\mathcal{T},x)$-admissible parenthesised word defined recursively by the following clauses:

- if $w = \underline{a}$, then $w^{\cdot x} = \underline{a}$,
- if $\mathcal{T} = \{\mathcal{T}_1\,(x_1x_2)\,\mathcal{T}_2\}$, $w = w_1w_2$, $w_i \in A(\mathcal{T}_i)$, $(\mathcal{T}_i, w_i) : X_i\ i = 1,2$, then
  - if $x \in X_1$, then $w^{\cdot x} = w_1^{\cdot x} w_2^{\cdot x_2}$,
  - if $x \in X_2$, then $w^{\cdot x} = w_2^{\cdot x} w_1^{\cdot x_1}$.

Observe that the successive commutations which transform w into the operadic word $w^{\cdot x}$ are witnessed in $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$ by the arrow term

$$\kappa_{(\mathcal{T},w,x)} : (\mathcal{T}, w) \to (\mathcal{T}, w^{\cdot x}),$$

defined recursively as follows:

- if $w = \underline{a}$, then $\kappa_{(\mathcal{T},w,x)} = 1_{(\mathcal{T},w)}$,
- if $\mathcal{T} = \{\mathcal{T}_1(x_1x_2)\mathcal{T}_2\}$, $w = w_1w_2$, $w_i \in A(\mathcal{T}_i)$ and $(\mathcal{T}_i, w_i) : X_i\ i = 1,2$, then

– if $x \in X_1$, then $\kappa_{(\mathcal{T},w,x)} = \kappa_{(\mathcal{T}_1,w_1,x)}\ {}_{x_1}\Box_{x_2}\ \kappa_{(\mathcal{T}_2,w_2,x_2)}$,
– if $x \in X_2$, then $\kappa_{(\mathcal{T},w,x)} = (\kappa_{(\mathcal{T}_2,w_2,x)}\ {}_{x_2}\Box_{x_1}\ \kappa_{(\mathcal{T}_1,w_1,x_1)}) \circ \gamma^{x_1,x_2}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}$.

Before we rigorously define the second reduction of arrow terms, we illustrate the idea behind it with a toy example.

Example 4.30. Consider the object term $(\mathcal{T},(\underline{a}\,\underline{b})\underline{c}) : X$, where $\mathcal{T}$ is defined as in Example 2.10. The arrow term

$$\beta^{x_i,y_{j_1};y_{j_2},z_l}_{(\mathcal{T}_1,\underline{a}),(\mathcal{T}_2,\underline{b}),(\mathcal{T}_3,\underline{c})} : (\mathcal{T},(\underline{a}\,\underline{b})\underline{c}) \to (\mathcal{T},\underline{a}(\underline{b}\,\underline{c}))$$

where $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are the subtrees of $\mathcal{T}$ determined by corollas a, b and c, respectively, is then well-typed and, by choosing $y_4 \in X$ (as we did in Example 4.28), we have

$$\mathtt{Red}_2(X,y_4)((\mathcal{T},(\underline{a}\,\underline{b})\underline{c})) = (\mathcal{T},y_4,(\underline{b}\,\underline{a})\underline{c}) \quad\text{and}\quad \mathtt{Red}_2(X,y_4)((\mathcal{T},\underline{a}(\underline{b}\,\underline{c}))) = (\mathcal{T},y_4,(\underline{b}\,\underline{c})\underline{a}).$$

For the two reductions of object terms, the arrow term

$$\theta^{y_2;y_3}_{(\mathcal{T}_2,y_4,\underline{b}),(\mathcal{T}_1,x_5,\underline{a}),(\mathcal{T}_3,z_1,\underline{c})} : (\mathcal{T},y_4,(\underline{b}\,\underline{a})\underline{c}) \to (\mathcal{T},y_4,(\underline{b}\,\underline{c})\underline{a})$$

is well-typed and it will be exactly the second reduction of $\beta^{x_i,y_{j_1};y_{j_2},z_l}_{(\mathcal{T}_1,\underline{a}),(\mathcal{T}_2,\underline{b}),(\mathcal{T}_3,\underline{c})}$. □

Formally, for an arrow term φ of $\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$, $\mathtt{Red}_2(X,x)(\varphi)$ is the arrow term defined recursively, as follows:

◇ $\mathtt{Red}_2(X,x)(1_{(\mathcal{T},w)}) = 1_{\mathtt{Red}_2(X,x)((\mathcal{T},w))}$,

◇ if $\varphi = \beta^{z,\underline{z};y,\underline{y}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}$, where $(\mathcal{T}_1,w_1) : X_i$, and

  – if $x \in X_1$, then $\mathtt{Red}_2(X,x)(\varphi) = \beta^{z;y}_{\mathtt{Red}_2(X_1,x)((\mathcal{T}_1,w_1)),\mathtt{Red}_2(X_2,\underline{z})((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_3,\underline{y})((\mathcal{T}_3,w_3))}$,
  – if $x \in X_2$, then $\mathtt{Red}_2(X,x)(\varphi) = \theta^{\underline{z};y}_{\mathtt{Red}_2(X_2,x)((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1)),\mathtt{Red}_2(X_3,\underline{y})((\mathcal{T}_3,w_3))}$,
  – if $x \in X_3$, then $\mathtt{Red}_2(X,x)(\varphi) = \beta^{\underline{y};\underline{z}\ -1}_{\mathtt{Red}_2(X_3,x)((\mathcal{T}_3,w_3)),\mathtt{Red}_2(X_2,y)((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))}$,

◇ if $\varphi = \beta^{z,\underline{z};y,\underline{y}\ -1}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}$, where $(\mathcal{T}_1,w_1) : X_i$, and

  – if $x \in X_1$, then $\mathtt{Red}_2(X,x)(\varphi) = \beta^{z;y\ -1}_{\mathtt{Red}_2(X_1,x)((\mathcal{T}_1,w_1)),\mathtt{Red}_2(X_2,\underline{z})((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_3,\underline{y})((\mathcal{T}_3,w_3))}$,
  – if $x \in X_2$, then $\mathtt{Red}_2(X,x)(\varphi) = \theta^{y;\underline{z}}_{\mathtt{Red}_2(X_2,x)((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_3,\underline{y})((\mathcal{T}_3,w_3)),\mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))}$,
  – if $x \in X_3$, then $\mathtt{Red}_2(X,x)(\varphi) = \beta^{\underline{y};\underline{z}}_{\mathtt{Red}_2(X_3,x)((\mathcal{T}_3,w_3)),\mathtt{Red}_2(X_2,y)((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))}$,

◇ if $\varphi = \gamma^{z,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}$, then $\mathtt{Red}_2(X,x)(\varphi) = 1_{\mathtt{Red}_2(X,x)((\{\mathcal{T}_1(zy)\mathcal{T}_2\},w_1w_2))}$

◇ if $\varphi = \varphi_2 \circ \varphi_1$, then $\mathtt{Red}_2(X,x)(\varphi) = \mathtt{Red}_2(X,x)(\varphi_2) \circ \mathtt{Red}_2(X,x)(\varphi_1)$,

◇ if $\varphi = \varphi_1\ {}_z\Box_y\ \varphi_2$, where $\varphi_1 : (\mathcal{T}_1,w_1) \to (\mathcal{T}_1',w_1')$, $\varphi_2 : (\mathcal{T}_2,w_2) \to (\mathcal{T}_2',w_2')$ and $(\mathcal{T}_i,w_i) : X_i$, then

  – if $x \in X_1$, then $\mathtt{Red}_2(X,x)(\varphi) = \mathtt{Red}_2(X_1,x)(\varphi_1)\ {}_z\Box_y\ \mathtt{Red}_2(X_2,y)(\varphi_2)$,
  – if $x \in X_2$, then $\mathtt{Red}_2(X,x)(\varphi) = \mathtt{Red}_2(X_2,x)(\varphi_2)\ {}_y\Box_z\ \mathtt{Red}_2(X_1,z)(\varphi_1)$.

Remark 4.31. *For an arrow term $\varphi : (\mathcal{T},w_1) \to (\mathcal{T},w_2)$ of $\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$, the type of $\mathtt{Red}_2(X,x)(\varphi)$ is*

$$\mathtt{Red}_2(X,x)(\varphi) : \mathtt{Red}_2(X,x)((\mathcal{T},w_1)) \to \mathtt{Red}_2(X,x)((\mathcal{T},w_2)).$$

Therefore, the second reduction of a pair of arrow terms of the same type *in $\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$ is a pair of arrow terms of the* same type *in $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$.*

The following theorem is the core of the coherence theorem. Intuitively, it says that the coherence of non-symmetric non-skeletal cyclic operads can be reduced to the coherence of non-symmetric non-skeletal operads[2]. As it will be clear from its proof, ($\beta\gamma$-hexagon) is the key coherence condition that makes this reduction possible.

Theorem 4.32. *For an arbitrary arrow term $\varphi : (\mathcal{T}, u) \to (\mathcal{T}, v)$ of $\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}$, the equality of interpretations*

$$\lfloor \kappa_{(\mathcal{T},v,x)} \rfloor_X \circ \lfloor \varphi \rfloor_X = \lceil \mathtt{Red}_2(X,x)(\varphi) \rceil_X \circ \lfloor \kappa_{(\mathcal{T},u,x)} \rfloor_X$$

holds in $\mathcal{C}(X)$.

Proof. By the definition of the interpretation function $\lfloor - \rfloor_X$, the equality of interpretations of arrow terms that we need to prove is

$$[\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\varphi)]_X = \lceil \mathtt{Red}_2(X,x)(\varphi) \rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},u,x)})]_X. \tag{4.1.3}$$

We proceed by induction on the structure of φ.

- If $\varphi = 1_{(\mathcal{T},w)}$, then

$$\begin{aligned}
[\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X \circ [\Delta_X(1_{(\mathcal{T},w)})]_X &= [\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X \\
&= 1_{(\mathcal{T},x,w\cdot x)} \circ [\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X \\
&= \lceil \mathtt{Red}_2(X,x)(1_{(\mathcal{T},w)}) \rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X.
\end{aligned}$$

- Suppose that $\varphi = \beta^{z,\underline{z};y,\underline{y}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}$, where $(\mathcal{T}_i, w_i) : X_i$.
  - If $x \in X_1$, then

$$\kappa_{(\mathcal{T},u,x)} = \big(\kappa_{(\mathcal{T}_1,w_1,x)}\ {}_{x}\square_{\underline{x}}\ \kappa_{(\mathcal{T}_2,w_2,\underline{z})}\big)\ {}_{y}\square_{\underline{y}}\ \kappa_{(\mathcal{T}_3,w_3,\underline{y})}$$

and

$$\kappa_{(\mathcal{T},v,x)} = \kappa_{(\mathcal{T}_1,w_1,x)}\ {}_{x}\square_{\underline{x}}\ \big(\kappa_{(\mathcal{T}_2,w_2,\underline{z})}\ {}_{y}\square_{\underline{y}}\ \kappa_{(\mathcal{T}_3,w_3,\underline{y})}\big).$$

Denote

$$\begin{array}{lll}
\kappa_1 = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,w_1,x)})]_{X_1} & \kappa_2 = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,w_2,\underline{z})})]_{X_2} & \kappa_3 = [\Delta_{X_3}(\kappa_{(\mathcal{T}_3,w_3,\underline{y})})]_{X_3} \\
f_1 = [\Delta_{X_1}((\mathcal{T}_1,w_1))]_{X_1} & f_2 = [\Delta_{X_2}((\mathcal{T}_2,w_2))]_{X_2} & f_3 = [\Delta_{X_3}((\mathcal{T}_3,w_3))]_{X_3} \\
f_1^\bullet = \lceil \mathtt{Red}_2(X_1,x)((\mathcal{T}_1,w_1)) \rceil_{X_1} & f_2^\bullet = \lceil \mathtt{Red}_2(X_2,\underline{z})(\mathcal{T}_2,w_2) \rceil_{X_2} & f_3^\bullet = \lceil \mathtt{Red}_2(X_3,\underline{y})(\mathcal{T}_3,w_3)) \rceil_{X_3}
\end{array}$$

The left-hand side and the right-hand side of (4.1.3) then correspond exactly to the top-right side and the left-bottom side, respectively, of the commuting diagram

$$\begin{tikzcd}
(f_1\ {}_{z}\circ_{\underline{z}}\ f_2)\ {}_{y}\circ_{\underline{y}}\ f_3 \arrow[r, "\beta^{z,\underline{z};y,\underline{y}}_{f_1,f_2,f_3}"] \arrow[d, "(\kappa_1\ {}_{z}\circ_{\underline{z}}\ \kappa_2)\ {}_{y}\circ_{\underline{y}}\ \kappa_3"'] & f_1\ {}_{z}\circ_{\underline{z}}\ (f_2\ {}_{y}\circ_{\underline{y}}\ f_3) \arrow[d, "\kappa_1\ {}_{z}\circ_{\underline{z}}\ (\kappa_2\ {}_{y}\circ_{\underline{y}}\ \kappa_3)"] \\
(f_1^\bullet\ {}_{z}\circ_{\underline{z}}\ f_2^\bullet)\ {}_{y}\circ_{\underline{y}}\ f_3^\bullet \arrow[r, "\beta^{z,\underline{z};y,\underline{y}}_{f_1^\bullet,f_2^\bullet,f_3^\bullet}"] & f_1^\bullet\ {}_{z}\circ_{\underline{z}}\ (f_2^\bullet\ {}_{y}\circ_{\underline{y}}\ f_3^\bullet)
\end{tikzcd}$$

which is a naturality diagram for β.

  - If $x \in X_2$, then

$$\kappa_{(\mathcal{T},u,x)} = \big((\kappa_{(\mathcal{T}_2,w_2,x)}\ {}_{\underline{z}}\square_{z}\ \kappa_{(\mathcal{T}_1,w_1,z)})\ {}_{y}\square_{\underline{y}}\ \kappa_{(\mathcal{T}_3,w_3,\underline{y})}\big) \circ \big(\gamma^{z,\underline{z}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}\ {}_{y}\square_{\underline{y}}\ 1_{(\mathcal{T}_3,w_3)}\big)$$

and

[2]Although it might seem that the syntax $\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}$ encodes canonical diagrams of non-symmetric categorified cyclic operads, this is not the case: non-symmetric cyclic operads still contain *cyclic actions* (see Remark 1.10).

$$\kappa_{(\mathcal{T},v,x)} = \big(\big(\kappa_{(\mathcal{T}_2,w_2,x)}\; {}_{y}\square_{\underline{y}}\; \kappa_{(\mathcal{T}_3,w_3,\underline{y})}\big)\; {}_{\underline{z}}\square_{z}\; \kappa_{(\mathcal{T}_1,w_1,z)}\big) \circ \big(\gamma^{z,\underline{z}}_{(\mathcal{T}_1,w_1),(\{\mathcal{T}_2(y\underline{y})\mathcal{T}_3\},w_2w_3)}\big).$$

Denote

$$\begin{array}{lll}
\kappa_1 = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,w_1,z)})]_{X_1} & \kappa_2 = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,w_2,x)})]_{X_2} & \kappa_3 = [\Delta_{X_3}(\kappa_{(\mathcal{T}_3,w_3,\underline{y})})]_{X_3} \\
f_1 = [\Delta_{X_1}((\mathcal{T}_1,w_1))]_{X_1} & f_2 = [\Delta_{X_2}((\mathcal{T}_2,w_2))]_{X_2} & f_3 = [\Delta_{X_3}((\mathcal{T}_3,w_3))]_{X_3} \\
f_1^{\bullet} = \lceil \mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))\rceil_{X_1} & f_2^{\bullet} = \lceil \mathtt{Red}_2(X_2,x)((\mathcal{T}_2,w_2))\rceil_{X_2} & f_3^{\bullet} = \lceil \mathtt{Red}_3(X_3,\underline{y})((\mathcal{T}_3,w_3))\rceil_{X_3}
\end{array}$$

The left-hand side and the right-hand side of (4.1.3) then correspond exactly to the top-right side and the left-bottom side, respectively, of the commuting diagram

$$\begin{tikzcd}
(f_1\; {}_{z}\circ_{\underline{z}}\; f_2)\; {}_{y}\circ_{\underline{y}}\; f_3 \arrow[r, "\beta^{z,\underline{z};y,\underline{y}}_{f_1,f_2,f_3}"] \arrow[d, "\gamma^{z,\underline{z}}_{f_1,f_2}\; {}_{y}\circ_{\underline{y}}\; 1_{f_3}"'] & f_1\; {}_{z}\circ_{\underline{z}}\; (f_2\; {}_{y}\circ_{\underline{y}}\; f_3) \arrow[d, "\gamma^{z,\underline{z}}_{f_1,f_2\; {}_{y}\circ_{\underline{y}}\; f_3}"] \\
(f_2\; {}_{\underline{z}}\circ_{z}\; f_1)\; {}_{y}\circ_{\underline{y}}\; f_3 \arrow[r, "\vartheta^{\underline{z},z;y,\underline{y}}_{f_2,f_1,f_3}"] \arrow[d, "(\kappa_2\; {}_{\underline{z}}\circ_{z}\; \kappa_1)\; {}_{y}\circ_{\underline{y}}\; \kappa_3"'] & (f_2\; {}_{y}\circ_{\underline{y}}\; f_3)\; {}_{\underline{z}}\circ_{z}\; f_1 \arrow[d, "(\kappa_2\; {}_{y}\circ_{\underline{y}}\; \kappa_3)\; {}_{\underline{z}}\circ_{z}\; \kappa_1"] \\
(f_2^{\bullet}\; {}_{\underline{z}}\circ_{z}\; f_1^{\bullet})\; {}_{y}\circ_{\underline{y}}\; f_3^{\bullet} \arrow[r, "\vartheta^{\underline{z},z;y,\underline{y}}_{f_2^{\bullet},f_1^{\bullet},f_3^{\bullet}}"] & (f_2^{\bullet}\; {}_{y}\circ_{\underline{y}}\; f_3^{\bullet})\; {}_{\underline{z}}\circ_{z}\; f_1^{\bullet}
\end{tikzcd}$$

in which the upper square commutes by the definition of the isomorphism ϑ (see (4.1.1)) and the lower square is a naturality diagram for ϑ.

– If $x \in X_3$, then

$$\begin{aligned}\kappa_{(\mathcal{T},u,x)} = {} & \big(\kappa_{(\mathcal{T}_3,w_3,x)}\; {}_{\underline{y}}\square_{y}\; \big(\kappa_{(\mathcal{T}_2,w_2,y)}\; {}_{\underline{z}}\square_{z}\; \kappa_{(\mathcal{T}_1,w_1,z)}\big)\big)\circ \\ & \big(1_{(\mathcal{T}_3,w_3)}\; {}_{\underline{y}}\square_{y}\; \gamma^{z,\underline{z}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}\big) \circ \gamma^{y,\underline{y}}_{(\{\mathcal{T}_1(z\underline{z})\mathcal{T}_2\},w_1w_2),(\mathcal{T}_3,w_3)}\end{aligned}$$

and

$$\begin{aligned}\kappa_{(\mathcal{T},v,x)} = {} & \big(\big(\kappa_{(\mathcal{T}_3,w_3,x)}\; {}_{\underline{y}}\square_{y}\; \kappa_{(\mathcal{T}_2,w_2,y)}\big)\; {}_{\underline{z}}\square_{z}\; \kappa_{(\mathcal{T}_1,w_1,z)}\big)\circ \\ & \big(\gamma_{(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}\; {}_{\underline{z}}\square_{z}\; 1_{(\mathcal{T}_1,w_1)}\big) \circ \gamma^{z,\underline{z}}_{(\mathcal{T}_1,w_1),(\{\mathcal{T}_2(y\underline{y})\mathcal{T}_3\},w_2w_3)}.\end{aligned}$$

Denote

$$\begin{array}{lll}
\kappa_1 = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,w_1,z)})]_{X_1} & \kappa_2 = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,w_2,y)})]_{X_2} & \kappa_3 = [\Delta_{X_3}(\kappa_{(\mathcal{T}_3,w_3,x)})]_{X_3} \\
f_1 = [\Delta_{X_1}((\mathcal{T}_1,w_1))]_{X_1} & f_2 = [\Delta_{X_2}((\mathcal{T}_2,w_2))]_{X_2} & f_3 = [\Delta_{X_3}((\mathcal{T}_3,w_3))]_{X_3} \\
f_1^{\bullet} = \lceil \mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))\rceil_{X_1} & f_2^{\bullet} = \lceil \mathtt{Red}_2(X_2,y)((\mathcal{T}_2,w_2))\rceil_{X_2} & f_3^{\bullet} = \lceil \mathtt{Red}_2(X_3,x)((\mathcal{T}_3,w_3))\rceil_{X_3}
\end{array}$$

The left-hand side and the right-hand side of (4.1.3) then correspond exactly to the top-right side and the left-bottom side, respectively, of the commuting diagram

$$\begin{tikzcd}
(f_1\; {}_{z}\circ_{\underline{z}}\; f_2)\; {}_{y}\circ_{\underline{y}}\; f_3 \arrow[r, "\beta^{z,\underline{z};y,\underline{y}}_{f_1,f_2,f_3}"] \arrow[d, "\gamma^{y,\underline{y}}_{f_1\; {}_{z}\circ_{\underline{z}}\; f_2,f_3}"'] & f_1\; {}_{z}\circ_{\underline{z}}\; (f_2\; {}_{y}\circ_{\underline{y}}\; f_3) \arrow[d, "\gamma^{z,\underline{z}}_{f_1,f_2\; {}_{y}\circ_{\underline{y}}\; f_3}"] \\
f_3\; {}_{\underline{y}}\circ_{y}\; (f_1\; {}_{z}\circ_{\underline{z}}\; f_2) \arrow[d, "1_{f_3}\; {}_{\underline{y}}\circ_{y}\; \gamma^{z,\underline{z}}_{f_1,f_2}"'] & (f_2\; {}_{y}\circ_{\underline{y}}\; f_3)\; {}_{\underline{z}}\circ_{z}\; f_1 \arrow[d, "\gamma^{y,\underline{y}}_{f_2,f_3}\; {}_{\underline{z}}\circ_{z}\; 1_{f_1}"] \\
f_3\; {}_{\underline{y}}\circ_{y}\; (f_2\; {}_{\underline{z}}\circ_{z}\; f_1) \arrow[r, "{\beta^{\underline{y},y;\underline{z},z}_{f_3,f_2,f_1}}^{-1}"] \arrow[d, "\kappa_3\; {}_{\underline{y}}\circ_{y}\; (\kappa_2\; {}_{\underline{z}}\circ_{z}\; \kappa_1)"'] & (f_3\; {}_{\underline{y}}\circ_{y}\; f_2)\; {}_{\underline{z}}\circ_{z}\; f_1 \arrow[d, "(\kappa_3\; {}_{\underline{y}}\circ_{y}\; \kappa_2)\; {}_{\underline{z}}\circ_{z}\; \kappa_1"] \\
f_3^{\bullet}\; {}_{\underline{y}}\circ_{y}\; (f_2^{\bullet}\; {}_{\underline{z}}\circ_{z}\; f_1^{\bullet}) \arrow[r, "{\beta^{\underline{y},y;\underline{z},z}_{f_3^{\bullet},f_2^{\bullet},f_1^{\bullet}}}^{-1}"] & (f_3^{\bullet}\; {}_{\underline{y}}\circ_{y}\; f_2^{\bullet})\; {}_{\underline{z}}\circ_{z}\; f_1^{\bullet}
\end{tikzcd}$$

in which the upper square commutes as an instance of ($\beta\gamma$-hexagon) and the bottom square commutes by the naturality of β^{-1}.

- The proof for the case $\varphi = \beta^{z,\underline{z};y,\underline{y}\,-1}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}$ follows directly from the previous item.
- Suppose now that $\varphi = \gamma^{z,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}$, where $(\mathcal{T}_i, w_i) : X_i$.
  - If $x \in X_1$, then
$$\kappa_{(\mathcal{T},u,x)} = \kappa_{(\mathcal{T}_1,w_1,x)}\, {}_z\square_y\, \kappa_{(\mathcal{T}_2,w_2,y)}$$
and
$$\kappa_{(\mathcal{T},v,x)} = \big(\kappa_{(\mathcal{T}_1,w_1,z)}\, {}_z\square_y\, \kappa_{(\mathcal{T}_2,w_2,x)}\big) \circ \gamma^{y,z}_{(\mathcal{T}_2,w_2),(\mathcal{T}_1,w_1)}.$$
Denote
$$\begin{array}{ll} \kappa_1 = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,w_1,x)})]_{X_1} & \kappa_2 = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,w_2,y)})]_{X_2} \\ f_1 = [\Delta_{X_1}((\mathcal{T}_1,w_1))]_{X_1} & f_2 = [\Delta_{X_2}((\mathcal{T}_2,w_2))]_{X_2} \\ f_1^{\bullet} = \lceil \mathtt{Red}_2(X_1,x)((\mathcal{T}_1,w_1))\rceil_{X_1} & f_2^{\bullet} = \lceil \mathtt{Red}_2(X_2,y)((\mathcal{T}_2,w_2))\rceil_{X_2} \end{array}$$
By (γ-involution), we then have
$$\begin{aligned} [\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\gamma^{z,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)})]_X &= \\ \big((\kappa_1\, {}_z\circ_y\, \kappa_2) \circ \gamma^{z,x}_{f_2,f_1}\big) \circ \gamma^{z,y}_{f_1,f_2} &= \\ \kappa_1\, {}_z\circ_y\, \kappa_2 &= \\ 1_{f_1^{\bullet}\, {}_z\circ_y\, f_2^{\bullet}} \circ (\kappa_1\, {}_z\circ_y\, \kappa_2) &= \\ \lceil \mathtt{Red}_2(X,x)(\gamma^{z,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)})\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},u,x)})]_X. \end{aligned}$$
  - The proof goes symmetrically if $x \in X_2$.
- If $\varphi = \varphi_2 \circ \varphi_1$, where $\varphi_1 : (\mathcal{T}, u) \to (\mathcal{T}, w)$ and $\varphi_2 : (\mathcal{T}, w) \to (\mathcal{T}, v)$, then, by the induction hypothesis for φ_1 and φ_2, we get
$$\begin{aligned} [\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\varphi_2 \circ \varphi_1)]_X &= \\ [\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\varphi_2)]_X \circ [\Delta_X(\varphi_1)]_X &= \\ \lceil \mathtt{Red}_2(X,x)(\varphi_2)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X \circ [\Delta_X(\varphi_1)]_X &= \\ \lceil \mathtt{Red}_2(X,x)(\varphi_2)\rceil_X \circ \lceil \mathtt{Red}_2(X,x)(\varphi_1)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X &= \\ \lceil \mathtt{Red}_2(X,x)(\varphi_2 \circ \varphi_1)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},u,x)})]_X. \end{aligned}$$
- Finally, suppose that $\varphi = \varphi_1\, {}_z\square_y\, \varphi_2$, where $\varphi_1 : (\mathcal{T}_1, u_1) \to (\mathcal{T}_1, v_1)$, $\varphi_2 : (\mathcal{T}_2, u_2) \to (\mathcal{T}_2, v_2)$, and $(\mathcal{T}_i, u_i) : X_i$.
  - If $x \in X_1$, then
$$\kappa_{(\mathcal{T},u,x)} = \kappa_{(\mathcal{T}_1,u_1,x)}\, {}_z\square_y\, \kappa_{(\mathcal{T}_2,u_2,y)}$$
and
$$\kappa_{(\mathcal{T},v,x)} = \kappa_{(\mathcal{T}_1,v_1,x)}\, {}_z\square_y\, \kappa_{(\mathcal{T}_2,v_2,y)}.$$
Denote
$$\begin{array}{ll} \kappa_{u_1} = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,u_1,x)})]_{X_1} & \kappa_{u_2} = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,u_2,y)})]_{X_2} \\ \kappa_{v_1} = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,v_1,x)})]_{X_1} & \kappa_{v_2} = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,v_2,y)})]_{X_2} \end{array}$$
By Remark 4.4.2(b) and the induction hypothesis for φ_1 and φ_2, we get

$$\begin{aligned}
[\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\varphi_1\, {}_z\square_y\, \varphi_2)]_X &= \\
(\kappa_{v_1}\, {}_z\circ_y\, \kappa_{v_2}) \circ ([\Delta_{X_1}(\varphi_1)]_{X_1}\, {}_z\circ_y\, [\Delta_{X_2}(\varphi_2)]_{X_2}) &= \\
(\kappa_{v_1} \circ [\Delta_{X_1}(\varphi_1)]_{X_1})\, {}_z\circ_y\, (\kappa_{v_2} \circ [\Delta_{X_2}(\varphi_2)]_{X_2}) &= \\
(\lceil \mathtt{Red}_2(X_1,x)(\varphi)\rceil_{X_1} \circ \kappa_{u_1})\, {}_z\circ_y\, (\lceil \mathtt{Red}_2(X_2,y)(\varphi_2)\rceil_{X_2} \circ \kappa_{u_2}) &= \\
(\lceil \mathtt{Red}_2(X_1,x)(\varphi)\rceil_{X_1}\, {}_z\circ_y\, \lceil \mathtt{Red}_2(X_2,y)(\varphi_2)\rceil_{X_2}) \circ (\kappa_{u_1}\, {}_z\circ_y\, \kappa_{u_2}) &= \\
\lceil \mathtt{Red}_2(X,x)(\varphi_1\, {}_z\square_y\, \varphi_2)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},u,x)})]_X. &
\end{aligned}$$

– If $x \in X_2$, then

$$\kappa_{(\mathcal{T},u,x)} = \left(\kappa_{(\mathcal{T}_2,u_2,x)}\, {}_y\square_z\, \kappa_{(\mathcal{T}_1,u_1,z)}\right) \circ \gamma^{z,y}_{(\mathcal{T}_1,u_1),(\mathcal{T}_2,u_2)}$$

and

$$\kappa_{(\mathcal{T},v,x)} = \left(\kappa_{(\mathcal{T}_2,v_2,x)}\, {}_y\square_z\, \kappa_{(\mathcal{T}_1,v_1,z)}\right) \circ \gamma^{z,y}_{(\mathcal{T}_1,v_1),(\mathcal{T}_2,v_2)}.$$

Denote

$$\begin{array}{ll}
\kappa_{u_1} = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,u_1,z)})]_{X_1} & \kappa_{u_2} = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,u_2,x)})]_{X_2} \\
\kappa_{v_1} = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,v_1,z)})]_{X_1} & \kappa_{v_2} = [\Delta_{X_2}([\kappa_{(\mathcal{T}_2,v_2,x)})]_{X_2} \\
f_{u_1} = [\Delta_{X_1}((\mathcal{T}_1,u_1))]_{X_1} & f_{u_2} = [\Delta_{X_2}((\mathcal{T}_2,u_2))]_{X_2} \\
f^{\bullet}_{u_1} = \lceil \mathtt{Red}_2(X_1,z)((\mathcal{T}_1,u_1))\rceil_{X_1} & f^{\bullet}_{u_2} = \lceil \mathtt{Red}_2(X_2,x)((\mathcal{T}_2,u_2))\rceil_{X_2} \\
f_{v_1} = [\Delta_{X_1}((\mathcal{T}_1,v_1))]_{X_1} & f_{v_2} = [\Delta_{X_2}((\mathcal{T}_2,v_2))]_{X_2} \\
f^{\bullet}_{v_1} = \lceil \mathtt{Red}_2(X_1,z)((\mathcal{T}_1,v_1))\rceil_{X_1} & f^{\bullet}_{v_2} = \lceil \mathtt{Red}_2(X_2,x)((\mathcal{T}_2,v_2))\rceil_{X_2}
\end{array}$$

By Remark 4.4.2(b), naturality of γ and the induction hypothesis for φ_1 and φ_2, we get

$$\begin{aligned}
[\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\varphi_1\, {}_z\square_y\, \varphi_2)]_X &= \\
(\kappa_{v_2}\, {}_y\circ_z\, \kappa_{v_1}) \circ \gamma^{z,y}_{f_{v_1},f_{v_2}} \circ ([\Delta_{X_1}(\varphi_1)]_{X_1}\, {}_z\circ_y\, [\Delta_{X_2}(\varphi_2)]_{X_2}) &= \\
\gamma^{z,y}_{f^{\bullet}_{v_1},f^{\bullet}_{v_2}} \circ (\kappa_{v_1}\, {}_z\circ_y\, \kappa_{v_2}) \circ ([\Delta_{X_1}(\varphi_1)]_{X_1}\, {}_z\circ_y\, [\Delta_{X_2}(\varphi_2)]_{X_2}) &= \\
\gamma^{z,y}_{f^{\bullet}_{v_1},f^{\bullet}_{v_2}} \circ ((\kappa_{v_1} \circ [\Delta_{X_1}(\varphi_1)]_{X_1})\, {}_z\circ_y\, (\kappa_{v_2} \circ [\Delta_{X_2}(\varphi_2)]_{X_2})) &= \\
\gamma^{z,y}_{f^{\bullet}_{v_1},f^{\bullet}_{v_2}} \circ ((\lceil \mathtt{Red}_2(X_1,z)(\varphi_1)\rceil_{X_1} \circ \kappa_{u_1})\, {}_z\circ_y\, (\lceil \mathtt{Red}_2(X_2,x)(\varphi_2)\rceil_{X_2} \circ \kappa_{u_2})) &= \\
((\lceil \mathtt{Red}_2(X_2,x)(\varphi_2)\rceil_{X_2} \circ \kappa_{u_2})\, {}_y\circ_z\, (\lceil \mathtt{Red}_2(X_1,z)(\varphi_1)\rceil_{X_1} \circ \kappa_{u_1})) \circ \gamma^{x,y}_{f_{u_1},f_{u_2}} &= \\
(\lceil \mathtt{Red}_2(X_2,x)(\varphi_2)\rceil_{X_2}\, {}_y\circ_z\, \lceil \mathtt{Red}_2(X_1,z)(\varphi_1)\rceil_{X_1}) \circ (\kappa_{u_2}\, {}_y\circ_z\, \kappa_{u_1}) \circ \gamma^{z,y}_{f_{u_1},f_{u_2}} &= \\
\lceil \mathtt{Red}_2(X,x)(\varphi_1\, {}_z\square_y\, \varphi_2)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},u,z)})]_X. &
\end{aligned}$$

■

The following result is a direct consequence of Theorem 4.32.

Corollary 4.33. *For arrow terms φ_1 and φ_2 of the same type in $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X)$, the equality*

$$\lfloor \varphi_1 \rfloor_X = \lfloor \varphi_2 \rfloor_X$$

follows from the equality

$$\lceil \mathtt{Red}_2(X,x)(\varphi_1)\rceil_X = \lceil \mathtt{Red}_2(X,x)(\varphi_2)\rceil_X.$$

4.1.5 The third reduction: establishing skeletality

Intuitively, in the third reduction we pass from the non-skeletal to the skeletal operadic framework. This will reduce the problem of commutation of all $\beta\vartheta$-diagrams of $\mathcal{C}(X)$ to the problem of commutation of all diagrams of canonical arrows of the skeletal non-symmetric categorified operad $\mathcal{O}_{\mathcal{C}}$, constructed from $\mathcal{C}$ in the appropriate way.

The skeletal non-symmetric categorified operad $\mathcal{O}_{\mathcal{C}}$

Starting from $\mathcal{C}$, we first define a *skeletal non-symmetric categorified operad* $\mathcal{O}_{\mathcal{C}} = \{\mathcal{O}_{\mathcal{C}}(n)\}_{n\in\mathbb{N}}$, i.e. a weak Cat-operad in the sense of [DP15], as follows.

- The objects of the category $\mathcal{O}_{\mathcal{C}}(n)$ are quadruplets (X, x, σ, f), where $|X| = n+1$, $x \in X$, $f \in \mathcal{C}(X)$ and $\sigma : [n] \to X\backslash\{x\}$ is a bijection (inducing a total order on $X\backslash\{x\}$).
- The morphisms of $\mathcal{O}_{\mathcal{C}}(n)[(X, x, \sigma, f), (X, x, \sigma, g)]$ are quadruplets (X, x, σ, φ), such that φ is a morphism of $\mathcal{C}(X)[f, g]$ (in particular, $\mathcal{O}_{\mathcal{C}}(n)[(X, x, \sigma, f), (Y, y, \tau, g)]$ is empty for $(X, x, \sigma) \neq (Y, y, \tau)$). The identity morphism for (X, x, σ, f) is $(X, x, \sigma, 1_f)$. The composition of morphisms is canonically induced from the composition of morphisms in $\mathcal{C}(X)$.
- The composition operation $\circ_i : \mathcal{O}_{\mathcal{C}}(n) \times \mathcal{O}_{\mathcal{C}}(m) \to \mathcal{O}_{\mathcal{C}}(n+m-1)$ on objects is defined by

$$(X, x, \sigma_1, f) \circ_i (Y, y, \sigma_2, g) = (X \cup Y\backslash\{y\}, x, \sigma, f\ {}_{\sigma_1(i)}{\circ}_y\ g),$$

  and on morphisms by

$$(X, x, \sigma_1, \varphi) \circ_i (Y, y, \sigma_2, \psi) = (X \cup Y\backslash\{y\}, x, \sigma, \varphi\ {}_{\sigma_1(i)}{\circ}_y\ \psi),$$

  where $\sigma : [n+m-1] \to X\backslash\{x\} \cup Y\backslash\{y\}$ is a bijection defined by

$$\sigma(j) = \begin{cases} \sigma_1(j) & \text{for } j \in \{1, \ldots, i-1\} \\ \sigma_2(j - i \cup 1) & \text{for } j \in \{i, \ldots, i+m-1\} \\ \sigma_1(j-m) & \text{for } j \in \{i+m, \ldots, n+m-1\}. \end{cases} \tag{4.1.4}$$

- For $\tilde{f} = (X, x, \sigma_1, f)$, $\tilde{g} = (Y, y, \sigma_2, g)$ and $\tilde{h} = (Z, z, \sigma_3, h)$, where $\sigma_1 : [n] \to X\backslash\{x\}$, $\sigma_2 : [m] \to Y\backslash\{y\}$ and $\sigma_3 : [k] \to Z\backslash\{z\}$, the components

$$\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}} : (\tilde{f} \circ_i \tilde{g}) \circ_j \tilde{h} \to \tilde{f} \circ_i (\tilde{g} \circ_j \tilde{h}) \quad \text{and} \quad \theta^{i;k}_{\tilde{f},\tilde{g},\tilde{h}} : (\tilde{f} \circ_i \tilde{g}) \circ_k \tilde{h} \to (\tilde{f} \circ_k \tilde{h}) \circ_i \tilde{h}$$

  of natural isomorphisms β and θ are distinguished among the morphisms of $\mathcal{O}_{\mathcal{C}}(n)$ as the quadruplets arising from the appropriate components of β and ϑ of $\mathcal{C}$, as follows:

$$\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}} = (X \cup Y\backslash\{y\} \cup Z\backslash\{z\}, x, \sigma, \beta^{\sigma_1(i),y;\sigma_2(j),z}_{f,g,h})$$

  and

$$\theta^{i;k}_{\tilde{f},\tilde{g},\tilde{h}} = (X \cup Y\backslash\{y\} \cup Z\backslash\{z\}, x, \sigma', \vartheta^{\sigma_1(i),y;\sigma_1(k),z}_{f,g,h}),$$

  where σ and σ' are the bijections induced in the appropriate way from σ_1, σ_2 and σ_3.

In the following lemma, we show that the structure $\mathcal{O}_{\mathcal{C}} = \{\mathcal{O}_{\mathcal{C}}(n)\}_{n\in\mathbb{N}}$ indeed verifies the axioms of weak Cat-operads given in [DP15, Section 7].

Lemma 4.34. *For an arbitrary $n \in \mathbb{N}$, the following equations hold in $\mathcal{O}_{\mathcal{C}}(n)$:*

1. *the categorical equations:*
   a) $\varphi \circ 1_{\tilde{f}} = \varphi = 1_{\tilde{g}} \circ \varphi$, *for* $\varphi : \tilde{f} \to \tilde{g}$,
   b) $(\varphi \circ \phi) \circ \psi = \varphi \circ (\phi \circ \psi)$,
2. *the bifunctoriality equations:*
   a) $1_{\tilde{f}} \circ_i 1_{\tilde{g}} = 1_{\tilde{f} \circ_i \tilde{g}}$,
   b) $(\varphi_2 \circ \varphi_1) \circ_i (\psi_2 \circ \psi_1) = (\varphi_2 \circ_i \psi_2) \circ (\varphi_1 \circ_i \psi_1)$,

3. *the naturality equations:*

   a) $\beta^{i;j}_{\tilde{f}_2,\tilde{g}_2,\tilde{h}_2}\circ((\varphi\circ_i\phi)\circ_j\psi)=(\varphi\circ_i(\phi\circ_j\psi))\circ\beta^{i;j}_{\tilde{f}_1,\tilde{g}_1,\tilde{h}_1}$,

   b) $\theta^{i;j}_{\tilde{f}_2,\tilde{g}_2,\tilde{h}_2}\circ((\varphi\circ_i\phi)\circ_j\psi)=((\varphi\circ_j\psi)\circ_i\phi)\circ\theta^{i;j}_{\tilde{f}_1,\tilde{g}_1,\tilde{h}_1}$,

   where $\varphi:\tilde{f}_1\rightarrow\tilde{f}_2$, $\phi:\tilde{g}_1\rightarrow\tilde{g}_2$ *and* $\psi:\tilde{h}_1\rightarrow\tilde{h}_2$,

5. *the equations concernig inverse isomorphisms:*

   a) ${\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}}^{-1}\circ\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}=1_{(\tilde{f}\circ_i\tilde{g})\circ_j\tilde{h}}$, $\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ{\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}}^{-1}=1_{\tilde{f}\circ_i(\tilde{g}\circ_j\tilde{h})}$,

   b) $\theta^{j;i}_{\tilde{f},\tilde{h},\tilde{g}}\circ\theta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}=1_{(\tilde{f}\circ_i\tilde{g})\circ_j\tilde{h}}$,

6. *the coherence conditions:*

   a) $(1_{\tilde{f}}\circ_i\beta^{j;l}_{\tilde{g},\tilde{h},\tilde{k}})\circ\beta^{i;l}_{\tilde{f},\tilde{g}\circ_j\tilde{h},\tilde{k}}\circ(\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ_l 1_{\tilde{k}})=\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}\circ_l\tilde{k}}\circ\beta^{j;l}_{\tilde{f}\circ_i\tilde{g},\tilde{h},\tilde{k}}$,

   b) $(1_{\tilde{f}}\circ_i\theta^{j;l}_{\tilde{g},\tilde{h},\tilde{k}})\circ\beta^{i;l}_{\tilde{f},\tilde{g}\circ_j\tilde{h},\tilde{k}}\circ(\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ_l 1_k)=\beta^{i;j}_{\tilde{f},\tilde{g}\circ_l\tilde{h},\tilde{k}}\circ(\beta^{i;l}_{\tilde{f},\tilde{g},\tilde{k}}\circ_j 1_{\tilde{h}})\circ\theta^{j;l}_{\tilde{f}\circ_i\tilde{g},\tilde{h},\tilde{k}}$,

   c) $\theta^{i;l}_{\tilde{f},\tilde{g}\circ_j\tilde{h},\tilde{k}}\circ(\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ_l 1_{\tilde{k}})=\beta^{i;j}_{\tilde{f}\circ_l\tilde{k},\tilde{g},\tilde{h}}\circ(\theta^{i;l}_{\tilde{f},\tilde{g},\tilde{k}}\circ_j 1_{\tilde{h}})\circ\theta^{j;l}_{\tilde{f}\circ_i\tilde{g},\tilde{h},\tilde{k}}$,

   d) $\theta^{i;j}_{\tilde{f}\circ_l\tilde{k},\tilde{g},\tilde{h}}\circ(\theta^{i;l}_{\tilde{f},\tilde{g},\tilde{k}}\circ_j 1_{\tilde{h}})\circ\theta^{j;l}_{\tilde{f}\circ_i\tilde{g},\tilde{h},\tilde{k}}=(\theta^{j;l}_{\tilde{f},\tilde{h},\tilde{k}}\circ_i 1_{\tilde{g}})\circ\theta^{i;l}_{\tilde{f}\circ_j\tilde{h},\tilde{g},\tilde{k}}\circ(\theta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ_l 1_{\tilde{k}})$.

Proof. The first two groups of equations, as well as the equation 3.(a), are verified straightforwardly by the corresponding groups of equations for $\mathcal{C}$, given in Remark 4.4. The equation 3.(b) follows by the naturality of ϑ (see (4.1.1)). The equation 5.(a) holds by the analogous equations for $\mathcal{C}$. The equation 5.(b) holds by Lemma 4.6. The equation 6.(a) holds by (β-pentagon), 6.(b) by ($\beta\gamma$-decagon) and Remark 4.5.(b), and 6.(c) and 6.(d) by Lemma 4.7. ■

"Skeletalisation" of the syntax $\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$: the syntax $_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$

In order to correctly apply the coherence result of [DP15], which is established for formal diagrams encoding the canonical diagrams of the skeletal non-symetric categorified operad $\mathcal{O}_{\mathcal{C}}$, we introduce the syntax of these diagrams. Intuitively, this syntax is a "skeletalisation" of the syntax $\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$.

Let $\mathcal{T}$ be an unrooted tree. Suppose that $FV(\mathcal{T})=X$ and let $x\in X$. For a corolla $a\in Cor(\mathcal{T})$, such that $|\mathtt{inp}_{(\mathcal{T},x)}(a)|=n$ (see the end of §4.1.4), we define the *set of skeletalisations of a (relative to $\mathcal{T}$ and x)* as

$$\Sigma_{(\mathcal{T},x)}(a)=\mathbf{Bij}[\,[n],\mathtt{inp}_{(\mathcal{T},x)}(a)\,].$$

We set

$$\Sigma(\mathcal{T},x)=\prod_{c\in Cor(\mathcal{T})}\Sigma_{(\mathcal{T},x)}(c).$$

We shall denote the elements of $\Sigma(\mathcal{T},x)$ with $\overrightarrow{\sigma}$.

Remark 4.35. *Notice that $\overrightarrow{\sigma_1}\in\Sigma(\mathcal{T}_1,x)$ and $\overrightarrow{\sigma_2}\in\Sigma(\mathcal{T}_2,y)$ determine "by concatenation" an element of $\overrightarrow{\sigma}\in\Sigma(\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x)$, and that, symmetrically, any $\overrightarrow{\sigma}\in\Sigma(\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x)$ can be "split" into $\overrightarrow{\sigma_1}\in\Sigma(\mathcal{T}_1,x)$ and $\overrightarrow{\sigma_2}\in\Sigma(\mathcal{T}_2,y)$. We shall denote this decomposition of $\overrightarrow{\sigma}$ with $\overrightarrow{\sigma_1\cdot\sigma_2}$.*

The *skeletalisation* $_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ *of the syntax* $\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ is the syntax obtained as follows.

The objects terms of $_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ are quadruplets $(\mathcal{T},x,\overrightarrow{\sigma},w)$, typed by the rule

$$\boxed{\dfrac{\mathcal{T}\in\underline{\mathtt{T}}^{+}_{\mathcal{C}}(X)\quad x\in X\quad \overrightarrow{\sigma}\in\Sigma(\mathcal{T},x)\quad w\in A(\mathcal{T},x)}{(\mathcal{T},x,\overrightarrow{\sigma},w):X\backslash\{x\}}}$$

The arrow terms of $_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ are obtained from raw terms

$$\chi ::= \begin{cases} 1_{(\mathcal{T},x,\vec{\sigma},w)} \mid \beta^{z;y}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)} \mid \beta^{z;y\;-1}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)} \\ \theta^{z;y}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)} \mid \chi\circ\chi \mid \chi\;{}_z\square_y\;\chi \end{cases}$$

by typing them as shown in Figure 4.8.

$$\frac{}{1_{(\mathcal{T},x,\vec{\sigma},w)} : (\mathcal{T},x,\vec{\sigma},w) \to (\mathcal{T},x,\vec{\sigma},w)}$$

$$\frac{\begin{array}{c}\mathcal{T}=\{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y\in FV(\mathcal{T}_2) \quad x\in X\cap FV(\mathcal{T}_1)\\ \vec{\sigma_1}\in\Sigma(\mathcal{T}_1,x) \quad \vec{\sigma_2}\in\Sigma(\mathcal{T}_2,\underline{z}) \quad \vec{\sigma_3}\in\Sigma(\mathcal{T}_3,\underline{y})\end{array}}{\beta^{z;y}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)} : (\mathcal{T},x,\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3},(w_1w_2)w_3\,) \to (\mathcal{T},x,\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3},w_1(w_2w_3)\,)}$$

$$\frac{\begin{array}{c}\mathcal{T}=\{\mathcal{T}_1\,(z\underline{z})\,\{\mathcal{T}_2\,(y\underline{y})\,\mathcal{T}_3\}\} \quad \underline{z}\in FV(\mathcal{T}_2) \quad x\in X\cap FV(\mathcal{T}_1)\\ \vec{\sigma_1}\in\Sigma(\mathcal{T}_1,x) \quad \vec{\sigma_2}\in\Sigma(\mathcal{T}_2,\underline{z}) \quad \vec{\sigma_3}\in\Sigma(\mathcal{T}_3,\underline{y})\end{array}}{\beta^{z;y\;-1}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)} : (\mathcal{T},x,\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3},w_1(w_2w_3)\,) \to (\mathcal{T},x,\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3},(w_1w_2)w_3\,)}$$

$$\frac{\begin{array}{c}\mathcal{T}=\{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y\in FV(\mathcal{T}_1) \quad x\in X\cap FV(\mathcal{T}_1)\\ \vec{\sigma_1}\in\Sigma(\mathcal{T}_1,x) \quad \vec{\sigma_2}\in\Sigma(\mathcal{T}_2,\underline{z}) \quad \vec{\sigma_3}\in\Sigma(\mathcal{T}_3,\underline{y})\end{array}}{\theta^{z;y}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)} : (\mathcal{T},x,\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3},(w_1w_2)w_3\,) \to (\mathcal{T},x,\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3},(w_1w_3)w_2\,)}$$

$$\frac{\chi_1 : (\mathcal{T},x,\vec{\sigma},w_1) \to (\mathcal{T},x,\vec{\sigma},w_2) \quad \chi_2 : (\mathcal{T},x,\vec{\sigma},w_2) \to (\mathcal{T},x,\vec{\sigma},w_3)}{\chi_2\circ\chi_1 : (\mathcal{T},x,\vec{\sigma},w_1) \to (\mathcal{T},x,\vec{\sigma},w_3)}$$

$$\frac{\chi_1 : (\mathcal{T}_1,x,\vec{\sigma_1},w_1) \to (\mathcal{T}_1,x,\vec{\sigma_1},w'_1) \quad \chi_2 : (\mathcal{T}_2,y,\vec{\sigma_2},w_2) \to (\mathcal{T}_2,y,\vec{\sigma_2},w'_2) \quad z\in FV(\mathcal{T}_1) \quad z\neq x}{\chi_1\;{}_z\square_y\;\chi_2 : (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x,\overrightarrow{\sigma_1\cdot\sigma_2},w_1w_2) \to (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x,\overrightarrow{\sigma_1\cdot\sigma_2},w'_1w'_2)}$$

FIGURE 4.8: Typing rules for the arrow terms of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$

As usual, we shall denote the class of object terms of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ with type X, together with the class of arrow terms whose types are pairs of object terms of type X, by ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}(X)$.

The interpretation of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ in $\mathcal{O}_{\mathcal{C}}$

In order to define the interpretation of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ in $\mathcal{O}_{\mathcal{C}}$, we first need to "order the inputs" of rooted trees figuring in object terms $(\mathcal{T},x,\vec{\sigma},w)$ of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$.

For an unrooted tree $\mathcal{T}$, a variable $x\in FV(\mathcal{T})$ and an element $\vec{\sigma}=(\sigma_1,\dots,\sigma_n)\in\Sigma_{(\mathcal{T},x)}$, the *total order of the inputs of $\mathcal{T}$ (relative to x) induced by $\vec{\sigma}$* is the bijection

$$\sigma : [\,|\mathtt{inp}_{(\mathcal{T},x)}(\mathcal{T})|\,] \to \mathtt{inp}_{(\mathcal{T},x)}(\mathcal{T})$$

defined recursively as follows:

- if $(\mathcal{T},x)=(\{a(x_1,\dots,x_n);id_X\},x_i)$, then $\sigma=\vec{\sigma}$,
- if $(\mathcal{T},x)=(\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x)$, $x\in FV(\mathcal{T}_1)$, $|\mathtt{inp}_{(\mathcal{T}_1,x)}(\mathcal{T}_1)|=n$, $|\mathtt{inp}_{(\mathcal{T}_2,y)}(\mathcal{T}_1)|=m$, $\sigma_1:[n]\to\mathtt{inp}_{(\mathcal{T}_1,x)}(\mathcal{T}_1)$ is the total order induced by $\vec{\sigma_1}\in\Sigma_{(\mathcal{T}_1,x)}$, $\sigma_2:[m]\to\mathtt{inp}_{(\mathcal{T}_2,y)}(\mathcal{T}_2)$ is the total order induced by $\vec{\sigma_2}\in\Sigma_{(\mathcal{T}_2,y)}$ and $\sigma_1(i)=z$, then

  $$\sigma : [n+m-1] \to FV(\mathcal{T})\backslash\{x\}$$

  is defined by (4.1.4).

The interpretation function

$$\lceil - \rceil^{\mathtt{sk}}_X : {}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}(X) \to \mathcal{O}_{\mathcal{C}}(|X|)$$

is defined recursively as follows:

- $\lceil\, (\{a(x_1,\dots,x_n); id_X\}, x_i, \vec{\sigma}, \underline{a})\, \rceil^{\mathtt{sk}}_{X\backslash\{x_i\}} = (\{x_1,\dots,x_n\}, x_i, \sigma, a)$,
- $\lceil(\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\}, x, \overrightarrow{\sigma_1\cdot\sigma_2}, w_1w_2)\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = \lceil(\mathcal{T}_1, x, \vec{\sigma_1}, w_1)\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}} {}_{\sigma_1^{-1}(z)}\!\circ\, \lceil(\mathcal{T}_2, y, \vec{\sigma_2}, w_2)\rceil^{\mathtt{sk}}_{X_2\backslash\{y\}}$,

and

- $\lceil 1_{(\mathcal{T},x,\vec{\sigma},w)}\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = 1_{\lceil(\mathcal{T},x,\vec{\sigma},w)\rceil^{\mathtt{sk}}_{X\backslash\{x\}}}$,
- $\lceil\beta^{z;y}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)}\rceil^{\mathtt{sk}}_{X\backslash\{x\}} =$
$$\beta^{\sigma_1^{-1}(z);\sigma_2^{-1}(y)}_{\lceil(\mathcal{T}_1,x,\vec{\sigma_1},w_1)\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}},\lceil(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2)\rceil^{\mathtt{sk}}_{X_2\backslash\{\underline{z}\}},\lceil(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)\rceil^{\mathtt{sk}}_{X_3\backslash\{\underline{y}\}}},$$
- $\lceil\beta^{z;y\;\;-1}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)}\rceil^{\mathtt{sk}}_{X\backslash\{x\}} =$
$$\beta^{\sigma_1^{-1}(z);\sigma_2^{-1}(y)\;\;-1}_{\lceil(\mathcal{T}_1,x,\vec{\sigma_1},w_1)\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}},\lceil(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2)\rceil^{\mathtt{sk}}_{X_2\backslash\{\underline{z}\}},\lceil(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)\rceil^{\mathtt{sk}}_{X_3\backslash\{\underline{y}\}}},$$
- $\lceil\theta^{z;y}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)}\rceil^{\mathtt{sk}}_{X\backslash\{x\}} =$
$$\theta^{\sigma_1^{-1}(z);\sigma_1^{-1}(y)}_{\lceil(\mathcal{T}_1,x,\vec{\sigma_1},w_1)\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}},\lceil(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2)\rceil^{\mathtt{sk}}_{X_2\backslash\{\underline{z}\}},\lceil(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)\rceil^{\mathtt{sk}}_{X_3\backslash\{\underline{y}\}}},$$
- $\lceil\chi_2\circ\chi_1\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = \lceil\chi_2\rceil^{\mathtt{sk}}_{X\backslash\{x\}}\circ\lceil\chi_1\rceil^{\mathtt{sk}}_{X\backslash\{x\}}$,
- $\lceil\chi_1\,{}_z\square_y\,\chi_2\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = \lceil\chi_1\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}}\,{}_{\sigma_1^{-1}(z)}\!\circ\,\lceil\chi_2\rceil^{\mathtt{sk}}_{X_2\backslash\{y\}}$,

where it is assumed that every total order σ (resp. σ_i) is induced by $\vec{\sigma}$ (resp. $\vec{\sigma_i}$).

The third reduction

In what follows, we shall denote with $\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}(X,x,\mathcal{T})$ the subclass of $\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}(X)$ determined by the rooted tree $(\mathcal{T},x)$ (i.e. by the object terms whose first two components are given by $(\mathcal{T},x)$ and by the arrow terms among them).

We define the family of *third reduction functions*

$$\mathtt{Red}_3(X,x,\mathcal{T},\vec{\sigma}) : \mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}(X,x,\mathcal{T}) \to {}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}_{\mathcal{C}}{}^{+}(X),$$

where $x\in X$, $\mathcal{T}$ is an unrooted tree such that $FV(\mathcal{T})=X$ and $\vec{\sigma}\in\Sigma_{(\mathcal{T},x)}$, as follows.

For object terms of $\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}(X,x,\mathcal{T})$, we set

$$\mathtt{Red}_3(X,x,\mathcal{T},\vec{\sigma})((\mathcal{T},x,w)) = (\mathcal{T},x,\vec{\sigma},w).$$

For an arrow term χ of $\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}(X,x,\mathcal{T})$, $\mathtt{Red}_3(X,x,\mathcal{T},\vec{\sigma})(\chi)$ is defined recursively as follows:

- $\mathtt{Red}_3(X,x,\mathcal{T},\vec{\sigma})(1_{(\mathcal{T},x,w)}) = 1_{\mathtt{Red}_3(X,x,\mathcal{T},\vec{\sigma})((\mathcal{T},x,w))}$,
- $\mathtt{Red}_3(X,x,\{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\},\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3})(\beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}) =$
$$\beta^{\sigma_1^{-1}(z);\sigma_2^{-1}(y)}_{(\mathcal{T}_1,x,\vec{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\vec{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\vec{\sigma_3},w_3)},$$

- ◇ $\mathtt{Red}_3(X,x,\{\mathcal{T}_1\,(z\underline{z})\,\{\mathcal{T}_2\,(y\underline{y})\,\mathcal{T}_3\}\},\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3})(\beta^{z;y\ \ -1}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}) =$
$$\beta^{\sigma_1^{-1}(z);\sigma_2^{-1}(y)\ \ -1}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)},$$
- ◇ $\mathtt{Red}_3(X,x,\{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\},\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3})(\theta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}) =$
$$\theta^{\sigma_1^{-1}(z);\sigma_1^{-1}(y)}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)},$$
- ◇ $\mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_2\circ\chi_1) = \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_2)\circ\mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_1)$,
- ◇ if $\chi = \chi_1\,{}_z\square_y\,\chi_2$, where $\chi_1 : (\mathcal{T}_1,x,w_1)\to(\mathcal{T}_1,x,w'_1)$ and $\chi_2 : (\mathcal{T}_2,y,w_2)\to(\mathcal{T}_2,y,w'_2)$, and if $\overrightarrow{\sigma_1}\in\Sigma_{(\mathcal{T}_1,x)}$ and $\overrightarrow{\sigma_2}\in\Sigma_{(\mathcal{T}_2,y)}$, then
$$\mathtt{Red}_3(X,x,\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},\overrightarrow{\sigma_1\cdot\sigma_2})(\chi_1\,{}_z\square_y\,\chi_2) = \mathtt{Red}_3(X_1,x,\mathcal{T}_1,\overrightarrow{\sigma_1})(\chi_1)\,{}_{\sigma_1^{-1}(z)}\square_{\sigma_2^{-1}(y)}\,\mathtt{Red}_3(X_2,y,\mathcal{T}_2,\overrightarrow{\sigma_2})(\chi_2).$$

Remark 4.36. *For the third reduction of an arrow term $\chi : (\mathcal{T},x,w_1)\to(\mathcal{T},x,w_2)$ of $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$, the type of $\mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi)$ is*

$$\mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi) : \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})((\mathcal{T},x,w_1))\to\mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})((\mathcal{T},x,w_2)).$$

Therefore, the third reduction of a pair of arrow terms of the same type *of $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$ is a pair of arrow terms of the* same type *of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}_{\underline{\mathcal{C}}}{}^+(X)$. Recall that the analogous properties hold for the first two reductions (see Lemma 4.23 and Remark 4.31).*

Theorem 4.37. *For an arbitrary object term $(\mathcal{T},x,w)$ and an arbitrary arrow term χ of $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$, the equalities*
$$\lceil\mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})((\mathcal{T},x,w))\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = (X,x,\sigma,\lceil(\mathcal{T},x,w)\rceil_X)$$
and
$$\lceil\mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi)\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = (X,x,\sigma,\lceil\chi\rceil_X)$$
hold, where the total order σ is induced from $\overrightarrow{\sigma}$.

Proof. We prove the first equality by induction on the proof of the $(\mathcal{T},x)$-admissibility of w.

- If $(\mathcal{T},x,w) = (\{a(x_1,\dots,x_n);id_X\},x_i,\underline{a})$, then
$$\begin{aligned}\lceil\mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})((\{a(x_1,\dots,x_n);id_X\},x_i,\underline{a}))\rceil^{\mathtt{sk}}_{X\backslash\{x_i\}} &= \\ \lceil(\{a(x_1,\dots,x_n);id_X\},x_i,\overrightarrow{\sigma},\underline{a})\rceil^{\mathtt{sk}}_{X\backslash\{x_i\}} &= \\ (\{x_1,\dots,x_n\},x_i,\sigma,a) &= \\ (\{x_1,\dots,x_n\},x_i,\sigma,\lceil(\{a(x_1,\dots,x_n);id_X\},x_i,\underline{a})\rceil_X)&\end{aligned}$$

- If $(\mathcal{T},x,w) = (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x,w_1w_2)$, then, by the induction hypothesis for $(\mathcal{T}_1,x,w_1) : X_1$ and $(\mathcal{T}_2,y,w_2) : X_2$, we get
$$\begin{aligned}\lceil\mathtt{Red}_3(X,x,\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},\overrightarrow{\sigma_1\cdot\sigma_2})((\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x,w_1w_2))\rceil^{\mathtt{sk}}_{X\backslash\{x\}} &= \\ \lceil\mathtt{Red}_3(X_1,x,\mathcal{T}_1,\overrightarrow{\sigma_1})((\mathcal{T}_1,x,w_1))\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}}\circ_{\sigma_1^{-1}(z)}\lceil\mathtt{Red}_3(X_2,y,\mathcal{T}_2,\overrightarrow{\sigma_2})((\mathcal{T}_2,y,w_2))\rceil^{\mathtt{sk}}_{X_2\backslash\{y\}} &= \\ (X_1,x,\sigma_1,\lceil(\mathcal{T}_1,x,w_1)\rceil_{X_1})\circ_{\sigma_1^{-1}(z)}(X_2,y,\sigma_2,\lceil(\mathcal{T}_2,y,w_2)\rceil_{X_2}) &= \\ (X,x,\sigma,\lceil(\mathcal{T}_1,x,w_1)\rceil_{X_1}\,{}_z\circ_y\,\lceil(\mathcal{T}_2,y,w_2)\rceil_{X_2}) &= \\ (X,x,\sigma,\lceil(\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x,w_1w_2)\rceil_X).&\end{aligned}$$

The second equality is proved by induction on the structure of χ, thanks to the first equality.

- For $\chi = 1_{(\mathcal{T},x,w)}$, we have

$$\begin{aligned}
\lceil \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(1_{(\mathcal{T},x,w)})\rceil^{\mathtt{sk}}_{X\backslash\{x\}} &= 1_{\lceil \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\mathcal{T},x,w)\rceil^{\mathtt{sk}}_{X\backslash\{x\}}}\\
&= 1_{(X,x,\sigma,\lceil(\mathcal{T},x,w)\rceil_X)}\\
&= (X,x,\sigma,\lceil 1_{(\mathcal{T},x,w)}\rceil_X).
\end{aligned}$$

- For $\chi = \beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}$, $\mathcal{T} = \{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\}$, $\overrightarrow{\sigma_1}\in\Sigma_{(\mathcal{T}_1,x)}$, $\overrightarrow{\sigma_2}\in\Sigma_{(\mathcal{T}_2,\underline{z})}$, $\overrightarrow{\sigma_3}\in\Sigma_{(\mathcal{T}_3,\underline{y})}$ and $\overrightarrow{\sigma} = \overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3}$ we have

$$\begin{aligned}
\lceil \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)})\rceil^{\mathtt{sk}}_{X\backslash\{x\}} &=\\
\lceil \beta^{z;y}_{\mathtt{Red}_3(X_1,x,\mathcal{T}_1,\overrightarrow{\sigma_1})((\mathcal{T}_1,x,w_1)),\mathtt{Red}_3(X_2,\underline{z},\mathcal{T}_2,\overrightarrow{\sigma_2})((\mathcal{T}_2,\underline{z},w_2)),\mathtt{Red}_3(X_3,\underline{y},\mathcal{T}_3,\overrightarrow{\sigma_3})((\mathcal{T}_3,\underline{y},w_3))}\rceil^{\mathtt{sk}}_{X\backslash\{x\}} &=\\
\beta^{\sigma_1^{-1}(z);\sigma_2^{-1}(y)}_{(X_1,x,\sigma_1,\lceil(\mathcal{T}_1,x,w_1)\rceil_{X_1}),(X_2,\underline{z},\sigma_2,\lceil(\mathcal{T}_2,\underline{z},w_2)\rceil_{X_2}),(X_3,\underline{y},\sigma_3,\lceil(\mathcal{T}_3,\underline{y},w_3)\rceil_{X_3})} &=\\
(X,x,\sigma,\beta^{z,\underline{z};y,\underline{y}}_{\lceil(\mathcal{T}_1,x,w_1)\rceil_{X_1},\lceil(\mathcal{T}_2,\underline{z},w_2)\rceil_{X_2},\lceil(\mathcal{T}_3,\underline{y},w_3)\rceil_{X_3}}) &=\\
(X,x,\sigma,\lceil\beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}\rceil_X).&
\end{aligned}$$

- We proceed similarly for $\chi = \beta^{z;y\;\;-1}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}$ and $\chi = \theta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)}$.

- If $\chi = \chi_2\circ\chi_1$, then, by the induction hypothesis for χ_1 and χ_2, we have

$$\begin{aligned}
\lceil \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_2\circ\chi_1)\rceil^{\mathtt{sk}}_{X\backslash\{x\}} &=\\
\lceil \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_2)\rceil^{\mathtt{sk}}_{X\backslash\{x\}}\circ\lceil \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_1)\rceil^{\mathtt{sk}}_{X\backslash\{x\}} &=\\
(X,x,\sigma,\lceil\chi_2\rceil_X)\circ(X,x,\sigma,\lceil\chi_1\rceil_X) &=\\
(X,x,\sigma,\lceil\chi_2\circ\chi_1\rceil_X).&
\end{aligned}$$

- If $\chi = \chi_1\,{}_z\square_y\,\chi_2$, where $\chi_1:(\mathcal{T}_1,x,w_1)\rightarrow(\mathcal{T}_1,x,w'_1)$ and $\chi_2:(\mathcal{T}_2,y,w_2)\rightarrow(\mathcal{T}_2,y,w'_2)$, then, for $\overrightarrow{\sigma} = \overrightarrow{\sigma_1\cdot\sigma_2}$, where $\overrightarrow{\sigma_1}\in\Sigma_{(\mathcal{T}_1,x)}$ and $\overrightarrow{\sigma_2}\in\Sigma_{(\mathcal{T}_2,y)}$, by the induction hypothesis for χ_1 and χ_2, we have

$$\begin{aligned}
\lceil \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_1\,{}_z\square_y\,\chi_2)\rceil^{\mathtt{sk}}_{X\backslash\{x\}} &=\\
\lceil \mathtt{Red}_3(X_1,x,\mathcal{T}_1,\overrightarrow{\sigma_1})(\chi_1)\;{}_{\sigma_1^{-1}(z)}\square_{\sigma_2^{-1}(y)}\;\mathtt{Red}_3(X_2,y,\mathcal{T}_2,\overrightarrow{\sigma_2})(\chi_2)\rceil^{\mathtt{sk}}_{X\backslash\{x\}} &=\\
\lceil \mathtt{Red}_3(X_1,x,\mathcal{T}_1,\overrightarrow{\sigma_1})(\chi_1)\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}}\circ_{\sigma_1^{-1}(z)}\lceil \mathtt{Red}_3(X_2,y,\mathcal{T}_2,\overrightarrow{\sigma_2})(\chi_2)\rceil^{\mathtt{sk}}_{X_2\backslash\{y\}} &=\\
(X_1,x,\sigma_1,\lceil\chi_1\rceil_{X_1})\circ_{\sigma_1^{-1}(z)}(X_2,y,\sigma_2,\lceil\chi_2\rceil_{X_2}) &=\\
(X,x,\sigma,\lceil\chi_1\,{}_z\square_y\,\chi_2\rceil_X).&
\end{aligned}$$

■

The following result is a direct consequence of Theorem 4.37.

Corollary 4.38. *For arrow terms χ_1 and χ_2 of the same type in $\mathtt{rT}^+_{\underline{\mathcal{C}}}(X)$, the equality*

$$\lceil\chi_1\rceil_X = \lceil\chi_1\rceil_X$$

follows from the equality

$$\lceil \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_1)\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = \lceil \mathtt{Red}_3(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_2)\rceil^{\mathtt{sk}}_{X\backslash\{x\}}.$$

4.1.6 The proof of the coherence theorem

We finally assemble the three reductions in the proof of the coherence theorem. The proof is outlined by the two invariance properties common for all three reductions: *by reducing a pair of arrow terms of the same type,*

1. *the result is always a pair of arrow terms of the same type, and*
2. *the equality of interpretations of the two resulting arrow terms implies the equality of the interpretations of the respective starting arrow terms.*

Coherence Theorem. *For any finite set X and for any pair of arrow terms $\Phi, \Psi : \mathcal{W}_1 \to \mathcal{W}_2$ of the same type in $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$, the equality $[[\Phi]]_X = [[\Psi]]_X$ holds in $\mathcal{C}(X)$.*

Proof. By Theorem 4.24 (first reduction), it is enough to prove the equality

$$[\mathtt{Red}_1(\Phi)]_X = [\mathtt{Red}_1(\Psi)]_X.$$

By Lemma 4.26 and Lemma 4.27, the problem translates to showing that

$$\lfloor \Delta_X^{-1}(\mathtt{Red}_1(\Phi)) \rfloor_X = \lfloor \Delta_X^{-1}(\mathtt{Red}_1(\Psi)) \rfloor_X.$$

By Corollary 4.33 (second reduction), this equality follows from the equality

$$\lceil \mathtt{Red}_2(X,x)(\Delta_X^{-1}(\mathtt{Red}_1(\Phi))) \rceil_X = \lceil \mathtt{Red}_2(X,x)(\Delta_X^{-1}(\mathtt{Red}_1(\Psi))) \rceil_X,$$

where $x \in X$ is arbitrary. By Corollary 4.38 (third reduction), the above equality holds if, in $\mathcal{O}_{\mathcal{C}}$, we have

$$\lceil \mathtt{Red}_3(X,x,\mathcal{T},\vec{\sigma})(\mathtt{Red}_2(X,x)(\Delta_X^{-1}(\mathtt{Red}_1(\Phi)))) \rceil^{\mathtt{sk}}_{X\backslash\{x\}} =$$
$$\lceil \mathtt{Red}_3(X,x,\mathcal{T},\vec{\sigma})(\mathtt{Red}_2(X,x)(\Delta_X^{-1}(\mathtt{Red}_1(\Psi)))) \rceil^{\mathtt{sk}}_{X\backslash\{x\}},$$

where $\mathcal{T}$ is the unrooted tree figuring in $\Delta_X^{-1}(\mathtt{Red}_1(W_s(\Phi)))$. Finally, the last equality holds by the coherence of $\mathcal{O}_{\mathcal{C}}$, established in [DP15]. ■

4.2 Categorified exchangeable-output cyclic operads

In the proof of Theorem 3.30, the equivalence between Definition 1.4 and Definition 3.25 has been worked out in detail. In this section, by adapting that equivalence to non-unital cyclic operads and by lifting it to the categorified setting, we set up the definition of *exchangeable-output non-skeletal categorified cyclic operads*. We finish the section by indicating how the definition of *exchangeable-output skeletal categorified cyclic operads* is obtained.

4.2.1 The exchangeable-output *non-skeletal* categorified cyclic operads

The categorification of Definition 3.25 is made by enriching the structure of a *categorified non-skeletal symmetric operad* $\mathcal{O}$ by *endofunctors* $D_x : \mathcal{O}(X) \to \mathcal{O}(X)$ that account for the exchange of the output with the input x, whose properties need to be such that the equivalence of Theorem 3.30 is not violated in the weakened setting. In other words, the decision whether some axiom of D_x should be weakened or not must respect the weakening made in passing from entries-only cyclic operads to their categorified version.

Before we give the resulting definition (with operadic units omitted), given that categorified operads of [DP15] are *skeletal* and *non-symmetric*, we first adapt their definition into a characterisation of categorified, *symmetric* and *non-skeletal* operads. As we did for categorified entries-only cyclic operads, we shall keep the equivariance axiom strict.

Definition 4.39. A *non-skeletal categorified operad* is a functor $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Cat}$, together with

- a family of bifunctors
$$\circ_x : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X\backslash\{x\} \cup Y),$$

indexed by arbitrary non-empty finite sets X and Y and element $x \in X$ such that $X\backslash\{x\} \cap Y = \emptyset$, subject to the equivariance axiom [EQ], and

- two natural isomorphisms, β and θ, called *sequential associativity* and *parallel associativity*, respectively, whose respective components

$$\beta^{x;y}_{f,g,h} : (f \circ_x g) \circ_y h \to f \circ_x (g \circ_y h) \quad \text{and} \quad \theta^{x;y}_{f,g,h} : (f \circ_x g) \circ_y h \to (f \circ_y h) \circ_x g\,,$$

are natural in f, g and h, and are subject to the following coherence conditions:

  – [θ-involution] $\theta^{y;x}_{f,h,g} \circ \theta^{x;y}_{f,g,h} = 1_{(f\circ_x g)\circ_y h}$,
  – [β-pentagon] $(1_f \circ_x \beta^{y;z}_{g,h,k}) \circ \beta^{x;z}_{f,g\circ_y h,k} \circ (\beta^{x;y}_{f,g,h} \circ_z 1_k) = \beta^{x;y}_{f,g,h\circ_z k} \circ \beta^{y;z}_{f\circ_x g,h,k}$,
  – [$\beta\theta$-hexagon] $(1_f \circ_x \theta^{y;z}_{g,h,k}) \circ \beta^{x;z}_{f,g\circ_y h,k} \circ (\beta^{x;y}_{f,g,h} \circ_z 1_k) = \beta^{x;y}_{f,g\circ_z h,k} \circ (\beta^{x;z}_{f,g,k} \circ_y 1_h) \circ \theta^{y;z}_{f\circ_x g,h,k}$,
  – [$\beta\theta$-pentagon] $\theta^{x;z}_{f,g\circ_y h,k} \circ (\beta^{x;y}_{f,g,h} \circ_z 1_k) = \beta^{x;y}_{f\circ_z k,g,h} \circ (\theta^{x;z}_{f,g,k} \circ_y 1_h) \circ \theta^{y;z}_{f\circ_x g,h,k}$,
  – [θ-hexagon] $\theta^{x;y}_{f\circ_z k,g,h} \circ (\theta^{x;z}_{f,g,k} \circ_y 1_h) \circ \theta^{y;z}_{f\circ_x g,h,k} = (\theta^{y;z}_{f,h,k} \circ_x 1_g) \circ \theta^{x;z}_{f\circ_y h,g,k} \circ (\theta^{x;y}_{f,g,h} \circ_z 1_k)$,
  – [$\beta\sigma$] if the equality $((f \circ_x g) \circ_y h)^\sigma = (f^{\sigma_1} \circ_{x'} g^{\sigma_2}) \circ_{y'} h^{\sigma_3}$ holds by [EQ], then

$$(\beta^{x;y}_{f,g,h})^\sigma = \beta^{x';y'}_{f^{\sigma_1},g^{\sigma_2},h^{\sigma_3}}\,,$$

  – [$\theta\sigma$] if the equality $((f \circ_x g) \circ_y h)^\sigma = (f^{\sigma_1} \circ_{x'} g^{\sigma_2}) \circ_{y'} h^{\sigma_3}$ holds by [EQ], then

$$(\theta^{x;y}_{f,g,h})^\sigma = \theta^{x';y'}_{f^{\sigma_1},g^{\sigma_2},h^{\sigma_3}}\,,$$

  – [EQ-mor] if the equality $(f \circ_x g)^\sigma = f^{\sigma_1} \circ_{x'} g^{\sigma_2}$ holds by [EQ], and if $\varphi : f \to f'$ and $\psi : g \to g'$, then

$$(\varphi \circ_x \psi)^\sigma = \varphi^{\sigma_1} \circ_{x'} \psi^{\sigma_2}.$$

□

Finally, here is the definition of non-skeletal categorified exchangeable-output cyclic operads. Recall from §3.3.2 that we write $D^{\mathcal{O}}_{xy}(f)$ for $D^{\mathcal{O}}_x(f)^\sigma$, where σ renames x to y.

Definition 4.40. A *non-skeletal categorified exchangeable-output cyclic operad* is a (non-skeletal) categorified operad $\mathcal{O}$, together with

- a family of endofunctors

$$D_x : \mathcal{O}(X) \to \mathcal{O}(X),$$

indexed by arbitrary finite sets X and elements $x \in X$, which are subject to the following axioms, in which f and g denote operadic operations and φ and ψ morphims between operadic operations:

[DIN] $D_x(D_x(f)) = f$ and $D_x(D_x(\varphi)) = \varphi$,

[DEQ] $D_x(f)^\sigma = D_{\sigma^{-1}(x)}(f^\sigma)$ and $D_x(\varphi)^\sigma = D_{\sigma^{-1}(x)}(\varphi^\sigma)$, where $\sigma : Y \to X$ is a bijection,

[DEX] $D_x(f)^\sigma = D_x(D_y(f))$ and $D_x(\varphi)^\sigma = D_x(D_y(\varphi))$, where $\sigma : X \to X$ exchanges x and y,

[DC1] $D_y(f \circ_x g) = D_y(f) \circ_x g$ and $D_y(\varphi \circ_x \psi) = D_y(\varphi) \circ_x \psi$, where $y \in X\backslash\{x\}$,

[$D\beta$] $D_z(\beta^{x;y}_{f,g,h}) = \beta^{x;y}_{D_z(f),g,h}$, where $f \in \mathcal{O}(X)$, $g \in \mathcal{O}(y)$, $h \in \mathcal{O}(Z)$, $x, z \in X$ and $y \in Y$,

[$D\theta$] $D_z(\theta^{x;y}_{f,g,h}) = \theta^{x;y}_{D_z(f),g,h}$, where $f \in \mathcal{O}(X)$, $g \in \mathcal{O}(y)$, $h \in \mathcal{O}(Z)$ and $x, y, z \in X$,

- a natural isomorphism δ, called the *exchange*, whose components

$$\delta^{y,x;v}_{f,g} : D_y(f \circ_x g) \to D_{yv}(g) \circ_v D_{xy}(f),$$

are natural in f and g, and are subject to the following coherence conditions:

- [$\delta\beta\theta$-square] for $f \in \mathcal{O}(X)$, $g \in \mathcal{O}(y)$, $h \in \mathcal{O}(Z)$, $x \in X$ and $y, z \in Y$, the following diagram commutes

$$\begin{tikzcd}
D_z((f \circ_x g) \circ_y h) \arrow[r, equal] \arrow[d, "D_z(\beta^{x;y}_{f,g,h})"'] & D_z(f \circ_x g) \circ_y h \arrow[d, "\delta^{z,x;v}_{f,g} \circ_y 1_h"] \\
D_z(f \circ_x (g \circ_y h)) \arrow[d, "\delta^{z,x;v}_{f,g\circ_y h}"'] & (D_{zv}(g) \circ_v D_{xz}(f)) \circ_y h \arrow[d, "\theta^{v;y}_{D_{zv}(g),D_{xz}(f),h}"] \\
D_{zv}(g \circ_y h) \circ_v D_{xz}(f) \arrow[r, equal] & (D_{zv}(g) \circ_y h) \circ_v D_{xz}(f)
\end{tikzcd}$$

- [$\delta\beta$-hexagon] for $f \in \mathcal{O}(X)$, $g \in \mathcal{O}(y)$, $h \in \mathcal{O}(Z)$, $x \in X$, $y \in Y$ and $z \in Z$, the following diagram commutes

$$\begin{tikzcd}
D_z((f \circ_x g) \circ_y h) \arrow[r, "\delta^{z,y;v}_{f\circ_x g,h}"] \arrow[d, "D_z(\beta^{x;y}_{f,g,h})"'] & D_{zv}(h) \circ_v D_{yz}(f \circ_x g) \arrow[d, "1_{D_{zv}(h)} \circ_v (\delta^{y,x;v}_{f,g})^{\tau}"] \\
D_z(f \circ_x (g \circ_y h)) \arrow[d, "\delta^{z,x;v}_{f,g\circ_y h}"'] & D_{zv}(h) \circ_v (D_{yv}(g) \circ_v D_{xy}(f))^{\sigma} \arrow[d, equal] \\
D_{zv}(g \circ_y h) \circ_v D_{xz}(f) \arrow[d, "(\delta^{z,y;v}_{g,h})^{\sigma} \circ_v 1_{D_{xz}(f)}"'] & D_{zv}(h) \circ_v (D_{yv}(g) \circ_v D_{xz}(f)) \arrow[d, "\beta^{v;x\,-1}_{D_{zv}(h),D_{yz}(g),D_{xz}(f)}"] \\
(D_{zv}(h) \circ_v D_{yz}(g))^{\sigma} \circ_v D_{xz}(f) \arrow[r, equal] & (D_{zv}(h) \circ_v D_{yv}(g)) \circ_v D_{xz}(f)
\end{tikzcd}$$

where σ renames z to v and τ renames y to z,

- [$D\delta$] for $f \in \mathcal{O}(X)$, $g \in \mathcal{O}(y)$ and $z \in Y$, the following diagram commutes

$$\begin{tikzcd}
D_z(D_z(f \circ_x g)) \arrow[r, "D_z(\delta^{z,x;v}_{f,g})"] \arrow[d, equal] & D_z(D_{zv}(g) \circ_v D_{xz}(f)) \arrow[d, "\delta^{z,v;u}_{D_{zv}(g),D_{xz}(f)}"] \\
f \circ_x g \arrow[r, equal] & D_{zu}(D_{xz}(f)) \circ_u D_{vz}(D_{zv}(g))
\end{tikzcd}$$

- [$\delta\sigma$] if the equality $(f \circ_x g)^{\sigma} = f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} g^{\sigma_2}$ holds by [EQ], then

$$(\delta^{z,x;v}_{f,g})^{\sigma} = \delta^{\sigma^{-1}(z),\sigma_1^{-1}(x);w}_{f^{\sigma_1},g^{\sigma_2}},$$

where $v \notin X\backslash\{x\} \cup Y\backslash\{z\}$ and $w \notin \sigma^{-1}[X\backslash\{x\} \cup Y\backslash\{z\}]$ are arbitrary variables. $\square$

Remark 4.41. *By comparing Definition 4.40 with Definition 3.25, one sees that the only axiom of D_x from Definition 3.25 that got weakened is* `[DC2]`. *Indeed, the proof of Theorem 3.30 testifies that all the axioms of D_x, except* `[DC2]`, *are proved by the functoriality and the equivariance of the corresponding entries-only cyclic operad, while the proof of* `[DC2]` *requires the axiom* `(CO)`. *Therefore, since* `(CO)` *gets weakened in passing from cyclic operads to categorified cyclic operads,* `[DC2]` *has to be weakened too.*

Remark 4.42. *Observe that, by* `[`$\delta\sigma$`]` *for $\sigma = id$, we have that*

$$\delta^{y,x;u}_{f,g} = \delta^{y,x;v}_{f,g}$$

for arbitrary variables $u, v \notin X\backslash\{x\} \cup Y\backslash\{z\}$.

We now lift the proof of Theorem 3.30 to the equivalence between the categorified versions of the (non-skeletal) entries-only and exchangeable-output of cyclic operads. The theorem that we establish has as a consequence the coherence of the latter notion. Henceforth, we shall restrict ourselves to constant-free categorified cyclic operads (as required by the proof of Theorem 3.30), the notion of which is obtained naturally from its decategorified variant, by replacing the empty set with the empty category (of operations).

Theorem 4.43. *Definition 4.1 (entries-only, categorified) and Definition 4.40 (exchangeable-output, categorified), restricted to constant-free categorified cyclic operads, are equivalent.*

Proof. We follow the lines of the proof of Theorem 3.30 and add the pieces of structures arising in the categorified framework.

Entries-only to Exchangeable-output. Let $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Cat}$ be an entries-only categorified cyclic operad. The functor $\mathcal{O}_{\mathcal{C}} : \mathbf{Bij}^{op} \to \mathbf{Cat}$, underlying the corresponding exchangeable-output categorified cyclic operad, is defined by (3.3.4). The partial composition operation $\circ_x : \mathcal{O}_{\mathcal{C}}(X) \times \mathcal{O}_{\mathcal{C}}(Y) \to \mathcal{O}_{\mathcal{C}}(X\backslash\{x\} \cup Y)$ is defined by (3.3.5). The action $D_x : \mathcal{O}_{\mathcal{C}}(X) \to \mathcal{O}_{\mathcal{C}}(X)$ is defined by (3.3.6).

The isomorphisms $\beta^{x,y}_{f,g,h}$ and $\theta^{x,y}_{f,g,h}$ are defined as follows. Let $f \in \mathcal{O}_{\mathcal{C}}(X)$, $g \in \mathcal{O}_{\mathcal{C}}(Y)$, $h \in \mathcal{O}_{\mathcal{C}}(Z)$ and $x \in X$. For $y \in Y$, we set

$$\beta^{x;y}_{f,g,h} = \beta^{x,*_Y;y,*_Z}_{f^\kappa,g,h},$$

where $\kappa : X \cup \{*_{X\backslash\{x\}\cup Y\backslash\{y\}\cup Z}\} \to X \cup \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x\}\cup Y\backslash\{y\}\cup Z}$. If $y \in X$, we set

$$\theta^{x,y}_{f,g,h} = \vartheta^{x,*_Y;y,*_Z}_{f^\kappa,g,h},$$

where $\kappa : X \cup \{*_{X\backslash\{x,y\}\cup Y\cup Z}\} \to X \cup \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x,y\}\cup Y\cup Z}$.

Finally, for $f \in \mathcal{O}_{\mathcal{C}}(X), g \in \mathcal{O}_{\mathcal{C}}(Y)$, $x \in X$ and $y \in Y$, we set

$$\delta^{y,x;v}_{f,g} = \gamma^{*_X,v}_{f^\kappa,g^\nu},$$

where $\kappa : X\backslash\{x\} \cup \{y\} \cup \{*_X\} \to X \cup \{*_X\}$ renames x to $*_X$ and $*_X$ to y, and $\nu : Y\backslash\{y\} \cup \{v\} \cup \{*_{X\backslash\{x\}\cup\{y\}}\} \to Y \cup \{*_Y\}$ renames $*_Y$ to v and y to $*_{X\backslash\{x\}\cup Y}$.

The coherence conditions of $\mathcal{O}_{\mathcal{C}}$ are verified as follows. We get `[`θ`-involution]` by Lemma 4.6, `[`β`-pentagon]` by `(`β`-pentagon)`, `[`$\beta\theta$`-hexagon]` by `(`$\beta\gamma$`-decagon)`, and `[`$\beta\theta$`-pentagon]` and `[`θ`-hexagon]` by Lemma 4.7. The coherence conditions `[`$\beta\sigma$`]`, `[`$\theta\sigma$`]`, `[EQ-mor]`, as well as `[`$D\beta$`]` and `[`$D\theta$`]`, hold by `(`$\beta\sigma$`)`, `(`$\gamma\sigma$`)` and `(EQ-mor)`. The equalities `[DIN]`, `[DEQ]`, `[DEX]` and `[DC1]` hold by the functoriality of $\mathcal{C}$ and `(EQ)`. The commutation of `[`$\delta\beta\theta$`-square]` follows by the definition of ϑ in $\mathcal{C}$ (see (4.1.1)). By redefining `[`$\delta\beta$`-hexagon]` in the language of the cyclic operad $\mathcal{C}$ (which is straightforward, but quite tedious), thanks to `(EQ)`, `(`$\beta\sigma$`)`, `(`$\gamma\sigma$`)` and `(EQ-mor)`, we get exactly an instance of `(`$\beta\gamma$`-hexagon)`. (We shall see how `(`$\beta\gamma$`-hexagon)` translates into `[`$\delta\beta$`-hexagon]` in the proof of the other transition below.) The condition `[`$D\delta$`]` follows by `(`$\gamma\sigma$`)`

and (γ-`involution`), and, finally, [$\delta\sigma$] follows by ($\gamma\sigma$).

Exchangeable-output to Entries-only. Let $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Cat}$ be an exchangeable-output categorified cyclic operad. In order to coerce the definition (3.3.7) into the definition of the functor $\mathcal{C}_{\mathcal{O}} : \mathbf{Bij}^{op} \to \mathbf{Cat}$, underlying the corresponding entries-only categorified cyclic operad, given that $\sum_{x\in X} \mathcal{O}(X\backslash\{x\})$ is now categorical sum in **Cat** (rather than disjoint union of sets), we simply extend the definition (3.3.3) of the equivalence relation $\approx$ to the category $\sum_{x\in X} \mathcal{O}(X\backslash\{x\})$. Unsurprisingly, to the family of generators of $\approx$ given by (3.3.3), we just add generators of the form

$$(x,\varphi) \approx (z, D_{zx}(\varphi)),$$

which will account for the equivalence classes of morphisms of

$$\mathcal{C}_{\mathcal{O}}(X) = \sum_{x\in X} \mathcal{O}(X\backslash\{x\})/_{\approx}.$$

The composition operation ${}_x\circ_y : \mathcal{C}_{\mathcal{O}}(X) \times \mathcal{C}_{\mathcal{O}}(Y) \to \mathcal{C}_{\mathcal{O}}(X\backslash\{x\}\cup Y\backslash\{y\})$ is defined by (3.3.8). In what follows, given that $\mathcal{O}$ (and, therefore, $\mathcal{C}_{\mathcal{O}}$) is constant-free, when calculating the composition $[(u,f)]_{\approx x}\circ_{\underline{x}}[(v,g)]_{\approx}$, we shall always assume that $u \neq x$ and $\underline{x} \neq v$. Furthermore, when considering the composite $([(u,f)]_{\approx x}\circ_{\underline{x}}[(v,g)]_{\approx})_y\circ_{\underline{y}}[(w,h)]_{\approx}$, where $g \in \mathcal{O}(Y)$, noticing that, in the "worst case", the set Y could be reduced to $\{\underline{x}\}$, we shall assume that $v = y$.

For the definition of $\beta^{x,\underline{x};y,\underline{y}}_{[(u,f)]_{\approx},[(v,g)]_{\approx},[(w,h)]_{\approx}}$, we calculate

$$([(u,f)]_{\approx}\,{}_x\circ_{\underline{x}}\,[(y,g)]_{\approx})\,{}_y\circ_{\underline{y}}\,[(w,h)]_{\approx} = [(u,(f\circ_x D_{\underline{x}y}(g))\circ_y D_{\underline{y}w}(h))]_{\approx}$$

and

$$[(u,f)]_{\approx}\,{}_x\circ_{\underline{x}}\,([(y,g)]_{\approx}\,{}_y\circ_{\underline{y}}\,[(w,h)]_{\approx}) = [(u,f\circ_x (D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h)))]_{\approx}$$

and we set

$$\beta^{x,\underline{x};y,\underline{y}}_{[(x,f)]_{\approx},[(y,g)]_{\approx},[(w,h)]_{\approx}} = [(u,\beta^{x;y}_{f,D_{\underline{x}y}(g),D_{\underline{y}w}(h)})]_{\approx}.$$

For the definition of $\gamma^{x,y}_{[(u,f)]_{\approx},[(v,g)]_{\approx}}$, we calculate

$$[(u,f)]_{\approx x}\circ_y[(v,g)]_{\approx} = [(u,f\circ_x D_{yv}(g))]_{\approx} \quad \text{and} \quad [(v,g)]_{\approx y}\circ_x[(u,f)]_{\approx} = [(v,g\circ_y D_{xu}(f))]_{\approx}.$$

Observe that, depending on the choice of the variable we take to be the common one for both classes (u or v), and by using [$\delta\sigma$], $\gamma^{x,y}_{[(u,f)]_{\approx},[(v,g)]_{\approx}}$ can be defined in two ways:

$$\gamma^{x,y}_{[(u,f)]_{\approx},[(v,g)]_{\approx}} = [(u,D_{uv}(\delta^{u,x;y}_{f,D_{yu}(g)}))]_{\approx} = [(v,\delta^{u,x;y}_{f,D_{yu}(g)})]_{\approx}.$$

We fix the definition

$$\gamma^{x,y}_{[(u,f)]_{\approx},[(v,g)]_{\approx}} = [(v,\delta^{u,x;y}_{f,D_{yu}(g)})]_{\approx}.$$

Therefore, in calculating an instance of the commutator, for the common variable of the source and the target we shall always choose the one of the target class.

As for the coherences of $\mathcal{C}_{\mathcal{O}}$, ($\beta$-`pentagon`) holds by [$\beta$-`pentagon`].

We show that ($\beta\gamma$-`hexagon`) holds by [$\delta\beta$-`hexagon`]. For the objects of ($\beta\gamma$-`hexagon`), we have

- $([(u,f)]_{\approx}\,{}_x\circ_{\underline{x}}\,[(y,g)]_{\approx})\,{}_y\circ_{\underline{y}}\,[(w,h)]_{\approx} = [(u,(f\circ_x D_{\underline{x}y}(g))\circ_y D_{\underline{y}w}(h))]_{\approx}$,
- $[(u,f)]_{\approx}\,{}_x\circ_{\underline{x}}\,([(y,g)]_{\approx}\,{}_y\circ_{\underline{y}}\,[(w,h)]_{\approx}) = [(u,f\circ_x (D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h)))]_{\approx}$,
- $([(y,g)]_{\approx}\,{}_y\circ_{\underline{y}}\,[(w,h)]_{\approx})\,{}_{\underline{x}}\circ_x\,[(u,f)]_{\approx} = [(w,D_{w\underline{x}}(D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h))\circ_{\underline{x}} D_{xu}(f))]_{\approx}$,
- $([(w,h)]_{\approx}\,{}_{\underline{y}}\circ_y\,[(y,g)]_{\approx})\,{}_{\underline{x}}\circ_x\,[(u,f)]_{\approx} = [(w,(h\circ_{\underline{y}} g)\circ_{\underline{x}} D_{xu}(f))]_{\approx}$,

- $[(w,h)]_{\approx}\ {}_{\underline{y}}\circ_y\ ([(y,g)]_{\approx}\ {}_{\underline{x}}\circ_x\ [(u,f)]_{\approx}) = [(w, h \circ_y (g \circ_{\underline{x}} D_{xu}(f)))]_{\approx}$,
- $([(y,g)]_{\approx}\ {}_{\underline{x}}\circ_x\ [(u,f)]_{\approx})\ {}_{y}\circ_{\underline{y}}\ [(w,h)]_{\approx} = [(u, D_{uy}(g \circ_{\underline{x}} D_{xu}(f)) \circ_y D_{\underline{y}w}(h))]_{\approx}$.

By replacing the representatives of the classes above with the ones whose first component is u, we get

- $([(u,f)]_{\approx}\ {}_{x}\circ_{\underline{x}}\ [(y,g)]_{\approx})\ {}_{y}\circ_{\underline{y}}\ [(w,h)]_{\approx} = [(u, (f \circ_x D_{\underline{x}y}(g)) \circ_y D_{\underline{y}w}(h))]_{\approx}$,
- $[(u,f)]_{\approx}\ {}_{x}\circ_{\underline{x}}\ ([(y,g)]_{\approx}\ {}_{y}\circ_{\underline{y}}\ [(w,h)]_{\approx}) = [(u, f \circ_x (D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)))]_{\approx}$,
- $([(y,g)]_{\approx}\ {}_{y}\circ_{\underline{y}}\ [(w,h)]_{\approx})\ {}_{\underline{x}}\circ_x\ [(u,f)]_{\approx} = [(u, D_{uw}(D_{w\underline{x}}(D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)) \circ_{\underline{x}} D_{xu}(f)))]_{\approx}$,
- $([(w,h)]_{\approx}\ {}_{\underline{y}}\circ_y\ [(y,g)]_{\approx})\ {}_{\underline{x}}\circ_x\ [(u,f)]_{\approx} = [(u, D_{uw}((h \circ_{\underline{y}} g) \circ_{\underline{x}} D_{xu}(f)))]_{\approx}$,
- $[(w,h)]_{\approx}\ {}_{\underline{y}}\circ_y\ ([(y,g)]_{\approx}\ {}_{\underline{x}}\circ_x\ [(u,f)]_{\approx}) = [(u, D_{uw}(h \circ_y (g \circ_{\underline{x}} D_{xu}(f))))]_{\approx}$,
- $([(y,g)]_{\approx}\ {}_{\underline{x}}\circ_x\ [(u,f)]_{\approx})\ {}_{y}\circ_{\underline{y}}\ [(w,h)]_{\approx} = [(u, D_{uy}(g \circ_{\underline{x}} D_{xu}(f)) \circ_y D_{\underline{y}w}(h))]_{\approx}$,

which determines the outer part of the following diagram of $\mathcal{O}$:

$$\begin{tikzcd}
(f \circ_x D_{\underline{x}y}(g)) \circ_y D_{\underline{y}w}(h) \arrow[r, "\beta^{x;y}_{f,D_{\underline{x}y}(g),D_{\underline{y}w}(h)}"] & f \circ_x (D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)) \arrow[rr, "(\delta^{u,\underline{x},x\,-1}_{D_{w\underline{x}}(D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h)),D_{xu}(f)})^{\tau}"] & & D_{uw}(D_{w\underline{x}}(D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)) \circ_{\underline{x}} D_{xu}(f)) \arrow[dl, "(\delta^{u,\underline{x};x}_{D_{w\underline{x}}(D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h)),D_{xu}(f)})^{\tau}"'] \arrow[ddd, "D_{uw}((\delta^{w,y;\underline{y}}_{D_{\underline{x}y}(g),D_{\underline{y}w}(h)})^{\kappa} \circ_{\underline{x}} 1_{D_{xu}(f)})"] \\
& & f \circ_x (D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)) \arrow[d, "1_f \circ_x (\delta^{\underline{x},\underline{y};y\,-1}_{h,g})^{\kappa^{-1}}"'] & \\
& & f \circ_x D_{\underline{x}w}(h \circ_{\underline{y}} g) \arrow[dr, "(\delta^{u,\underline{x};x\,-1}_{h\circ_{\underline{y}}g,D_{xu}(f)})^{\tau}"'] & \\
D_{uy}(g \circ_{\underline{x}} D_{xu}(f)) \circ_y D_{\underline{y}w}(h) \arrow[uuu, "(\delta^{u,\underline{x};x}_{g,D_{xu}(f)})^{\nu} \circ_y 1_{D_{\underline{y}w}(h)}"] \arrow[r, "(\delta^{u,\underline{y};y\,-1}_{h,g\circ_{\underline{x}}D_{xu}(f)})^{\tau}"'] & D_{uw}(h \circ_{\underline{y}} (g \circ_{\underline{x}} D_{xu}(f))) & & D_{uw}((h \circ_{\underline{y}} g) \circ_{\underline{x}} D_{xu}(f)) \arrow[ll, "D_{uw}(\beta^{\underline{y};\underline{x}}_{h,g,D_{xu}(f)})"']
\end{tikzcd}$$

where τ renames u to w, κ renames w to $\underline{x}$ and ν renames u to y, and in which the square on the right commutes by naturality of δ. The equality

$$(\delta^{u,\underline{x},x}_{D_{w\underline{x}}(D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h)),D_{xu}(f)})^{\tau} \circ (\delta^{u,\underline{x},x\,-1}_{D_{w\underline{x}}(D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h)),D_{xu}(f)})^{\tau} = 1_{f\circ_x(D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h))},$$

together with `[`$\delta\sigma$`]`, turns the diagram above into the following instance of `[`$\delta\beta$`-hexagon]`:

$$\begin{tikzcd}
(f \circ_x D_{\underline{x}y}(g)) \circ_y D_{\underline{y}w}(h) \arrow[r, "\beta^{x;y}_{f,D_{\underline{x}y}(g),D_{\underline{y}w}(h)}"] & f \circ_x (D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)) \arrow[r, "1_f \circ_x (\delta^{\underline{x},\underline{y};y\,-1}_{h,g})^{\kappa^{-1}}"] & f \circ_x D_{\underline{x}w}(h \circ_{\underline{y}} g) \arrow[d, "\delta^{w,\underline{x};x\,-1}_{h\circ_{\underline{y}}g,D_{xw}(f)}"] \\
D_{uy}(g \circ_{\underline{x}} D_{xu}(f)) \circ_y D_{\underline{y}w}(h) \arrow[u, "(\delta^{u,\underline{x};x}_{g,D_{xu}(f)})^{\nu} \circ_y 1_{D_{\underline{y}w}(h)}"] \arrow[r, "\delta^{w,\underline{y};y\,-1}_{h,g\circ_{\underline{x}}D_{xw}(f)}"'] & D_{uw}(h \circ_{\underline{y}} (g \circ_{\underline{x}} D_{xu}(f))) & D_{uw}((h \circ_{\underline{y}} g) \circ_{\underline{x}} D_{xu}(f)) \arrow[l, "D_{uw}(\beta^{\underline{y};\underline{x}}_{h,g,D_{xu}(f)})"']
\end{tikzcd}$$

For `(`$\beta\gamma$`-decagon)`, we use `[`$\beta\theta$`-hexagon]` together with `[`$\delta\beta\theta$`-square]`. We illustrate the proof by showing that the composition of the top three morphisms of `(`$\beta\gamma$`-decagon)` is exactly an instance of the isomorphsm θ. The four objects figuring in the composition of those three morphisms are:

- $((([(u,f)]_{\approx}\ {}_{x}\circ_{\underline{x}}\ [(y,g)]_{\approx})\ {}_{y}\circ_{\underline{y}}\ [(z,h)]_{\approx})\ {}_{z}\circ_{\underline{z}}\ [(w,k)]_{\approx} = [(u, ((f \circ_x D_{\underline{x}y}(g)) \circ_y D_{\underline{y}z}h) \circ_z D_{\underline{z}w}(k))]_{\approx}$,
- $([(z,h)]_{\approx}\ {}_{\underline{y}}\circ_y\ ([(u,f)]_{\approx}\ {}_{x}\circ_{\underline{x}}\ [(y,g)]_{\approx}))\ {}_{z}\circ_{\underline{z}}\ [(w,k)]_{\approx} = [(u, D_{uz}(h \circ_{\underline{y}} D_{yu}(f \circ_x D_{\underline{x}y}(g))) \circ_z D_{\underline{z}w}(k))]_{\approx}$,

- $[(z,h)]_{\approx}\ {}_{\underline{y}}{\circ}_{y}\ (([[(u,f)]_{\approx}\ {}_{x}{\circ}_{\underline{x}}\ [(y,g)]_{\approx})\ {}_{z}{\circ}_{\underline{z}}\ [(w,k)]_{\approx}) = [(u, D_{uz}(h \circ_{\underline{y}} (D_{yu}(f \circ_x D_{\underline{x}y}(g)) \circ_z D_{\underline{z}w}(k))))]_{\approx}$,
- $(([[(u,f)]_{\approx}\ {}_{x}{\circ}_{\underline{x}}\ [(y,g)]_{\approx})\ {}_{z}{\circ}_{\underline{z}}\ [(w,k)]_{\approx})\ {}_{y}{\circ}_{\underline{y}}\ [(z,h)]_{\approx} = [(u, ((f \circ_x D_{\underline{x}y}(y)) \circ_z D_{\underline{z}w}(k)) \circ_y D_{\underline{y}z}(h))]_{\approx}$.

For the definitions of the top three morphisms of ($\beta\gamma$-decagon), we have:

- $\gamma^{y,\underline{y}}_{[(u,f)]_{\approx}\ {}_{x}\circ_{\underline{x}}\ [(y,g)]_{\approx},[(z,h)]_{\approx}}\ {}_{z}{\circ}_{\underline{z}}\ 1_{[(w,k)]_{\approx}} = (\delta^{u,\underline{y};y\ -1}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g))})^{\sigma} \circ_z 1_{D_{\underline{z}w}(k)}$,
- $\beta^{\underline{y},y;z,\underline{z}}_{[(z,h)]_{\approx},[(u,f)]_{\approx}\ {}_{x}\circ_{\underline{x}}\ [(y,g)]_{\approx},[(w,k)]_{\approx}} = D_u(\beta^{\underline{y};z}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g)),D_{\underline{z}w}(k)})^{\sigma}$, and
- $\gamma^{\underline{y},y}_{[(z,h)]_{\approx},([(u,f)]_{\approx}\ {}_{x}\circ_{\underline{x}}\ [(y,g)]_{\approx})\ {}_{z}\circ_{\underline{z}}\ [(w,k)]_{\approx}} = (\delta^{u,\underline{y};y}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g))\circ_z D_{\underline{z}w}(k)})^{\sigma}$,

where σ renames u to z. By [$\beta\sigma$], [$\delta\sigma$] and [DEQ] , we get

- $(\delta^{u,\underline{y};y\ -1}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g))})^{\sigma} \circ_z 1_{D_{\underline{z}w}(k)} = \delta^{z,\underline{y};y\ -1}_{h,D_z(f\circ_x D_z(g^{\tau}))} \circ_z 1_{D_{\underline{z}w}(k)}$,
- $D_u(\beta^{\underline{y};z}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g)),D_{\underline{z}w}(k)})^{\sigma} = D_z(\beta^{\underline{y};z}_{h,D_z(f\circ_x D_z(g^{\tau})),D_{\underline{z}w}(k)})$, and
- $(\delta^{u,\underline{y};y}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g))\circ_z D_{\underline{z}w}(k)})^{\sigma} = \delta^{z,\underline{y};y}_{h,D_z(f\circ_x D_z(g^{\tau}))\circ_z D_{\underline{z}w}(k))}$

where τ renames $\underline{x}$ to z. Finally, by [$\delta\beta\theta$-square] and [$\theta\sigma$], we get that

$$\delta^{z,\underline{y};y\ -1}_{h,D_z(f\circ_x D_z(g^{\tau}))} \circ_z 1_{D_{\underline{z}w}(k)} \circ D_z(\beta^{\underline{y};z}_{h,D_z(f\circ_x D_z(g^{\tau})),D_{\underline{z}w}(k)}) \circ (\delta^{z,\underline{y};y\ -1}_{h,D_z(f\circ_x D_z(g^{\tau}))} \circ_z 1_{D_{\underline{z}w}(k)}) = \theta^{z;y}_{f\circ_x g,h,k}.$$

For (γ-involution), observe that $\gamma^{y,x}_{[(v,g)]_{\approx},[(u,f)]_{\approx}}$ is defined exactly in a way which makes the composition $\gamma^{y,x}_{[(v,g)]_{\approx},[(u,f)]_{\approx}} \circ \gamma^{x,y}_{[(u,f)]_{\approx},[(v,g)]_{\approx}}$ figure favorably in [$D\delta$].

The isomorphism of categorified cyclic operads $\mathcal{C}$ and $\mathcal{C}_{\mathcal{O}_{\mathcal{C}}}$ (and $\mathcal{O}$ and $\mathcal{O}_{\mathcal{C}_{\mathcal{O}}}$). The two isomorphisms are easily defined from their corresponding decategorified versions in the proof of Theorem 3.30. ∎

4.2.2 The exchangeable-output *skeletal* categorified cyclic operads

Given that the skeletal exchangeable-output characterisation of cyclic operads is arguably most commonly seen in the literature (cf. [Mar08, Proposition 42.]), we round up this work by indicating that the categorification of this notion is made straightforwardly by translating Definition 4.40 to the skeletal setting. The coherence of the obtained notion follows by lifting to the categorified setting the equivalence of non-skeletal and skeletal operads, established in [MSS02, Theorem 1.61], extended naturally so that it also includes endofunctors $D_x : \mathcal{O}(X) \to \mathcal{O}(X)$ (for non-skeletal operads) and $D_i : \mathcal{O}(n) \to \mathcal{O}(n)$ (for skeletal operads). In Appendix A, we describe in detail the equivalence of [MSS02, Theorem 1.61] (whose lifting, which we omit, is then easily derived).

Future Work

In this thesis, we examined three different frameworks for the general theory of cyclic operads. As cyclic operads make a relatively young concept of abstract algebra, the motivation was to set up theoretical grounds of various flavours for that concept, which would then be starting points for exploring perspectives of cyclic operads and finding their embodiments in different fields of mathematics, as well as for bringing the knowledge from those fields back to abstract algebra, in order to propose solutions to problems inherent to operad theory. Therefore, the ideas for future investigation can be summarised as the application of the theoretical results established by the thesis in finding the place and the use of cyclic operads "in nature".

From the syntactic point of view, it would be natural to apply a syntactic approach similar to the one of Chapter 2 to other variations of operads (like modular operads of [GK98]), and to investigate the adjustments to be made in the case where symmetries (other than cyclic permutations) are not present.

Prompted by the algebraic definitions of Chapter 3, it would be interesting to rephrase the construction of invariant bilinear forms on algebras over a cyclic operad $\mathcal{C}$ of [GK95] in the algebraic setting. More generally, Chapter 3 provides a "recipe" for redefining other variations and generalisations of operads internally to the category of species, which aims at making their investigation modular, so as not to repeat analysis and constructions of the same kind for each variation of the notion of operad. Also, it would be interesting to see what would a Lie bracket $[S,T] = S\blacktriangle T - T\blacktriangle S$ mean in the context of cyclic operads, but in order to understand this, we first have to give a meaning to the operation $-$ used in its definition, which indicates that we need to replace set-theoretical species of structures with $\mathbb{S}$-modules.

As for the categorified setting, given the context in which operads and cyclic operads have emerged, the main questions that arise are the question of exhibiting categorified cyclic operads of Chapter 4 "in nature" and the question of determining whether they could be of some use in applications in non-commutative geometry and algebraic topology. At this moment, there are two tangible directions of investigation, one for each of these two questions.

The first one is about the construction of a skeletal categorified cyclic operad $\mathcal{C} : \Sigma^{op} \to \mathbf{Cat}$ in the form of a bicategory of generalised profunctors (i.e. distributors) of [Bén73].

Task 1. *Given a category $\mathbf{D}$ with enough colimits, which is, moreover, equipped with a duality $(-)^* : \mathbf{D}^{op} \to \mathbf{D}$, a categorified cyclic operad $\mathcal{C} : \Sigma^{op} \to \mathbf{Cat}$ can be exhibited by defining the category of n-ary operations of $\mathcal{C}$ as $\mathcal{C}(n) = [D^n, \mathbf{Set}]$, which can be seen as a generalised form of a profunctor, and by using a generalised form of the composition of profunctors for defining the partial composition of $\mathcal{C}$. For example, if $F \in \mathcal{C}(3)$ and $G \in \mathcal{C}(2)$, the operation*

$$F\, {}_2{\circ}_1\, G : D^3 \to \mathbf{Set}$$

is defined by

$$(F\, {}_2{\circ}_1\, G)(a,c,e) = \int^{b,d} F(a,b,c) \times G(d,e) \times [b,d],$$

where $[-,-] : D^{op} \times D^{op} \to \mathbf{Set}$ is defined by $[x,y] = D[x,y^*]$. One must then define the associator and commutatior isomorphisms and look for the possible conditions that D has to satisfy in order to be able to verify the corresponding coherence conditions.

The second direction is about applications of categorified (cyclic) operads in proving the Koszulness of coloured (cyclic) operads which encode (cyclic) operads, by exhibiting their Gröbner bases.

Task 2. *Show that the coherence theorem for categorified (cyclic) operads can be adapted to a generalisation of the Diamond Lemma for coloured (cyclic) operads encoding (cyclic) operads. This task requires to consider the notion of coloured (cyclic) operad as a structure for which, apart from operadic compositions, the action of the symmetric group also makes a part of the defining structure. In other words, the action of the symmetric group is not introduced implicitly by building the definition over an* $\mathbb{S}$*-module, but rather by starting from an* $\mathbb{N}$*-module and then modeling it explicitly, as an additional set of generators (accompanied by the appropriate set of relations), as done in [DV15, Definition 5]. In this way, the candidate for the Gröbner basis is given by the set of tree-polynomials corresponding to canonical isomorphisms of a categorified (cyclic) operad defined in the skeletal manner, for which the equivariance axiom is additionally weakened.*

Finally, motivated by the combinatorial approach to operadic polytopes made in [DP11], the following task arises naturally in the setting of categorified cyclic operads.

Task 3. *Find polytopes which describe coherences and higher coherences of categorified cyclic operads. An initial analysis of this problem, made for the 3-dimensional setting by considering all the 2-dimensional coherence diagrams arising from a linear tree with five nodes (which would correspond to the facets of the appropriate polytope), shows that the number of such coherence facets is* rather large.

Appendix A

Skeletal vs. Non-skeletal operadic framework

There are no true winners of this battle, since the two frameworks are equivalent for **Set**-based (cyclic) operads. In general, they are equivalent under fairly weak assumptions: the base symmetric monoidal category $(\mathbf{C}, \otimes)$ should have small colimits, and, for an arbitrary object X of $\mathbf{C}$, the endofunctor $X \otimes (-)$ should preserve colimits. However, this does not mean that they are *equally good* for a particular matter of investigation. Before we summarise why *our* winner is the non-skeletal framework, we give details of the equivalence of the two approaches.

A.1 The equivalence

Let $\mathcal{O} : \mathbf{\Sigma}^{op} \to \mathbf{Set}$ be a skeletal operad, defined as in [LV12, 5.3.7] (only without units). We shall write $\mathcal{O}(n)$ instead of $\mathcal{O}([n])$, and we shall denote the operadic composition operations of $\mathcal{O}$ with $\diamond_i$. Quoting [LV12, 5.3.7], the equivariance of $\mathcal{O}$ is given by the following two relations:

[EQ1] For any $\sigma \in \mathbb{S}_m$, we have

$$f \diamond_i g^{\sigma} = (f \diamond_i g)^{\sigma'},$$

where $\sigma' \in \mathbb{S}_{n+m-1}$ is the permutation which acts by the identity, except on the block $\{i, \ldots, i+m-1\}$, on which it acts by σ.

[EQ2] For any $\sigma \in \mathbb{S}_n$, we have

$$f^{\sigma} \diamond_i g = (f \diamond_{\sigma(i)} g)^{\sigma''},$$

where $\sigma'' \in \mathbb{S}_{n+m-1}$ is acting like σ on the block $\{1, \ldots, n+m-1\} \backslash \{i, \ldots, i+m-1\}$ with values in $\{1, \ldots, n+m-1\} \backslash \{\sigma(i), \ldots \sigma(i)+m-1\}$ and identically on the block $\{i, \ldots, i+m-1\}$ with values in $\{\sigma(i), \ldots \sigma(i)+m-1\}$.

As the definition of a non-skeletal operad, we fix Definition 1.1 (again, without units). For a non-skeletal operad $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Set}$, we shall continue to denote with $\circ_x$ its operadic composition morphisms.

Theorem A.1. [LV12, 5.3.7] *and Definition 1.1 are equivalent definitions of symmetric operads.*

Proof. We make the transitions between the structures specified by the two definitions and we show that they determine an isomorphism of operads.

Skeletal to Non-skeletal. Let $\mathcal{O} : \mathbf{\Sigma}^{op} \to \mathbf{Set}$ be a skeletal operad. The functor underlying the corresponding non-skeletal operad $\mathcal{O}_{ns} : \mathbf{Bij}^{op} \to \mathbf{Set}$ is defined as

$$\mathcal{O}_{ns}(X) = \{[(f, \varphi_X)]_{\approx} \mid f \in \mathcal{O}(n) \text{ and } \varphi_X : X \to [n] \ (\varphi_X \text{ bijective})\},$$

where $|X| = n$ and $\approx$ is the smallest equivalence relation generated by

$$(f, \varphi_X) \approx (f^{\varphi_X \circ \psi_X^{-1}}, \psi_X), \tag{A.1.1}$$

where $\psi_X : X \to [n]$. For a bijection $\sigma : Y \to X$, the function $\mathcal{O}_{ns}(\sigma) : \mathcal{O}_{ns}(X) \to \mathcal{O}_{ns}(Y)$ is defined by

$$[(f, \varphi_X)]_{\approx}^{\sigma} = [(f, \varphi_X \circ \sigma)]_{\approx}.$$

The composition operation $\circ_x : \mathcal{O}_{ns}(X) \times \mathcal{O}_{ns}(Y) \to \mathcal{O}_{ns}(X \backslash \{x\} \cup Y)$ is defined as follows. Let $[(f, \varphi_X)]_{\approx} \in \mathcal{O}_{ns}(X)$ and $[(g, \varphi_Y)]_{\approx} \in \mathcal{O}_{ns}(Y)$, where $|X| = n$ and $|Y| = m$, and let $x \in X$. We set

$$[(f, \varphi_X)]_{\approx} \circ_x [(g, \varphi_Y)]_{\approx} = [(f \diamond_{\varphi_X(x)} g, \varphi_Z)]_{\approx},$$

where $Z = X \backslash \{x\} \cup Y$ and

$$\varphi_Z(v) = \begin{cases} \varphi_X(v) & \text{for all } v \in X \text{ such that } \varphi_X(v) < \varphi_X(x) \\ \varphi_Y(v) + \varphi_X(x) - 1 & \text{for all } v \in Y \\ \varphi_X(v) + m - 1 & \text{for all } v \in X \text{ such that } \varphi_X(v) > \varphi_X(x). \end{cases} \tag{A.1.2}$$

The equivariance axiom `[EQ]` of $\mathcal{O}$ ensures that the definition of $\circ_x$ does not depend on the choice of φ_X and φ_Y. Indeed, if $\varphi'_X : X \to [n]$ and $\varphi'_Y : Y \to [m]$ are different from φ_X and ψ_X, respectively, then

$$\begin{aligned} [(f^{\varphi_X \circ {\varphi'_X}^{-1}}, \varphi'_X)]_{\approx} \circ_x [(g^{\varphi_Y \circ {\varphi'_Y}^{-1}}, \varphi'_Y)]_{\approx} &= [(f^{\varphi_X \circ {\varphi'_X}^{-1}} \diamond_{\varphi'_X(x)} g^{\varphi_Y \circ {\varphi'_Y}^{-1}}, \varphi'_Z)]_{\approx} \\ &= [((f \diamond_{\varphi_X(x)} g)^{\varphi_Z \circ {\varphi'_Z}^{-1}}, \varphi'_Z)]_{\approx} \\ &= [(f \diamond_{\varphi_X(x)} g, \varphi_Z)]_{\approx} \\ &= [(f, \varphi_X)]_{\approx} \circ_x [(g, \varphi_Y)]_{\approx}. \end{aligned}$$

It is easily seen that the associativity axioms `[A1]` and `[A2]` of $\mathcal{O}$ ensure the corresponding associativity axioms of $\mathcal{O}_{ns}$.

Finally, we show that the equivariance axiom `[EQ]` of $\mathcal{O}_{ns}$ comes "for free". Let $[(f, \varphi_X)]_{\approx} \in \mathcal{O}_{ns}(X)$, $[(g, \varphi_Y)]_{\approx} \in \mathcal{O}_{ns}(Y)$ and $x \in X$. Then, for an arbitrary bijection $\sigma : U \to X \backslash \{x\} \cup Y$ and bijections $\sigma_1 : V_1 \to X$ and $\sigma_2 : V_2 \to Y$, such that $\sigma = \sigma_1|^{X \backslash \{x\}} \cup \sigma_2|^{Y}$, we have

$$\begin{aligned} ([(f, \varphi_X)]_{\approx} \circ_x [(g, \varphi_Y)]_{\approx})^{\sigma} &= [(f \diamond_{\varphi_X(x)} g, \varphi_Z \circ \sigma)] \\ &= [(f, \varphi_X \circ \sigma_1)]_{\approx} \circ_{\sigma_1^{-1}(x)} [(g, \varphi_Y \circ \sigma_2)]_{\approx} \\ &= [(f, \varphi_X)]_{\approx}^{\sigma_1} \circ_{\sigma_1^{-1}(x)} [(g, \varphi_Y)]_{\approx}^{\sigma_2}. \end{aligned}$$

the key being that $\varphi_Z \circ \sigma$ coincides with the bijection built in the obvious way from $\varphi_X \circ \sigma_1$ and $\varphi_Y \circ \sigma_2$.

Non-skeletal to Skeletal. Let now $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Set}$ be a non-skeletal operad. The functor underlying the corresponding skeletal operad $\mathcal{O}_s : \mathbf{\Sigma}^{op} \to \mathbf{Set}$ is defined by

$$\mathcal{O}_s(n) = \mathcal{O}(n) \quad \text{and} \quad \mathcal{O}_s(\sigma) = \mathcal{O}(\sigma).$$

The composition operation $\diamond_i : \mathcal{O}_s(n) \times \mathcal{O}_s(m) \to \mathcal{O}_s(n+m-1)$ is defined by

$$f \diamond_i g = f^{\sigma_1} \circ_i g^{\sigma_2},$$

where $\sigma_1 : \{1, \ldots, i\} + \{i+m, \ldots, n+m-1\} \to [n]$ and $\sigma_2 : \{i, i+1, \ldots, i+m-1\} \to [m]$ are defined as follows:

$$\sigma_1(j) = \begin{cases} j & \text{if } j \in \{1, \ldots, i\} \\ j - m + 1 & \text{if } j \in \{i+m, \ldots, n+m-1\} \end{cases} \quad \text{and} \quad \sigma_2(k) = k - i + 1. \tag{A.1.3}$$

Therefore, $\mathcal{O}_s : \mathbf{\Sigma}^{op} \to \mathbf{Set}$ is defined by restricting the data of $\mathcal{O}$ in the natural way.

Notice that the proof of associativity of $\mathcal{O}_s$ requires both the associativity and the equivariance of $\mathcal{O}$.

Here is the proof of `[EQ1]` of $\mathcal{O}_s$. Let $f \in \mathcal{O}_s(n)$, $g \in \mathcal{O}_s(m)$, $1 \leq i \leq n$ and let $\tau_2 : [m] \to [m]$ be a permutation. We then have

$$\begin{aligned} f \diamond_i g^{\tau_2} &= f^{\sigma_1} \circ_i (g^{\tau_2})^{\sigma_2} \\ &= f^{\sigma_1} \circ_i g^{\tau_2 \circ \sigma_2} \\ &= f^{\sigma_1} \circ_i g^{\sigma_2 \circ (\sigma_2^{-1} \circ \tau_2 \circ \sigma_2)} \\ &= f^{\sigma_1} \circ_i (g^{\sigma_2})^{\sigma_2^{-1} \circ \tau_2 \circ \sigma_2} \\ &= (f^{\sigma_1} \circ_i g^{\sigma_2})^{\tau} \\ &= (f \diamond_i g)^{\tau}, \end{aligned}$$

where $\sigma_1 : \{1, \ldots, i\} \cup \{i+m, \ldots, n+m-1\} \to [n]$ and $\sigma_2 : \{i, i+1, \ldots, i+m-1\} \to [m]$ are defined as in (A.1.3), $\tau : [n+m-1] \to [n+m-1]$ is defined as

$$\tau = id_{\{1,\ldots,i-1\} \cup \{i+m,\ldots,n+m-1\}} \cup (\sigma_2^{-1} \circ \tau_2 \circ \sigma_2),$$

and the equality

$$f^{\sigma_1} \circ_i (g^{\sigma_2})^{\sigma_2^{-1} \circ \tau_2 \circ \sigma_2} = (f^{\sigma_1} \circ_i g^{\sigma_2})^{\tau}$$

holds by the equivariance axiom `[EQ]` of $\mathcal{O}$.

We now prove `[EQ2]`. If $\tau_1 : [n] \to [n]$ is a permutation, we have

$$f^{\tau_1} \diamond_i g = (f^{\tau_1})^{\sigma_1} \circ_i g^{\sigma_2} = f^{\tau_1 \circ \sigma_1} \circ_i g^{\sigma_2},$$

where $\sigma_1 : \{1, \ldots, i\} \cup \{i+m, \ldots, n+m-1\} \to [n]$ and $\sigma_2 : \{i, i+1, \ldots, i+m-1\} \to [m]$ are defined by (A.1.3). On the other hand, we have

$$f \diamond_{\tau_1(i)} g = f^{\kappa_1} \circ_{\tau_1(i)} g^{\kappa_2},$$

where $\kappa_1 : \{1, \ldots, \tau_1(i)\} \cup \{\tau_1(i)+m, \ldots, n+m-1\} \to [n]$ and $\kappa_2 : \{\tau_1(i), \tau_1(i)+1, \ldots, \tau_1(i)+m-1\} \to [m]$ are defined as

$$\kappa_1(j) = \begin{cases} j & \text{if } j \in \{1, \ldots, \tau_1(i)\} \\ j-m+1 & \text{if } j \in \{\tau_1(i)+m, \ldots, n+m-1\} \end{cases} \quad \text{and} \quad \kappa_2(k) = k - \tau_1(i) + 1.$$

Let

$$\tau_2 : \{i, i+1, \ldots, i+m-1\} \to \{\tau_1(i), \tau_1(i)+1, \ldots, \tau_1(i)+m-1\}$$

be a bijection defined as

$$\tau_2 = \kappa_2^{-1} \circ \sigma_2.$$

We then have

$$\begin{aligned} f^{\tau_1 \circ \sigma_1} \circ_i g^{\sigma_2} &= f^{\kappa_1 \circ (\kappa_1^{-1} \circ \tau_1 \circ \sigma_1)} \circ_i g^{\kappa_2 \circ \tau_2} \\ &= (f^{\kappa_1})^{\kappa_1^{-1} \circ \tau_1 \circ \sigma_1} \circ_i (g^{\kappa_2})^{\tau_2} \\ &= (f^{\kappa_1} \circ_{(\kappa_1^{-1} \circ \tau_1 \circ \sigma_1)(i)} g^{\kappa_2})^{\tau} \\ &= (f^{\kappa_1} \circ_{\tau_1(i)} g^{\kappa_2})^{\tau} \\ &= (f \diamond_{\tau_1(i)} g)^{\tau}, \end{aligned}$$

where $\tau : [n+m-1] \to [n+m-1]$ is defined as

$$\tau = (\kappa_1^{-1} \circ \tau_1 \circ \sigma_1)|^{\{1,\ldots,\tau_1(i)-1\} \cup \{\tau_1(i)+m,\ldots,n+m-1\}} \cup \tau_2,$$

and the equality

$$(f^{\kappa_1})^{\kappa_1^{-1}\circ\tau_1\circ\sigma_1} \circ_i (g^{\kappa_2})^{\tau_2} = (f^{\kappa_1} \circ_{(\kappa_1^{-1}\circ\tau_1\circ\sigma_1)(i)} g^{\kappa_2})^{\tau}$$

holds by the equivariance axiom [EQ] of $\mathcal{O}$. Notice that, in the proofs of both equations, τ is acting exactly like specified by the equivariance of $\mathcal{O}_s$. This makes the equivariance established.

The isomorphism of operads $\mathcal{O}$ and $(\mathcal{O}_{ns})_s$. The bijection $\phi_{[n]}$ between the sets $\mathcal{O}(n)$ and $(\mathcal{O}_{ns})_s(n) = \mathcal{O}_{ns}(n)$ is defined by

$$\phi_{[n]} : f \mapsto [(f, id_{[n]})]_{\approx}.$$

The remaining of the (skeletal) operad structure transfers via $\phi_{[n]}$ as follows:

$$\phi_{[n]}(f^{\sigma}) = [(f^{\sigma}, id)]_{\approx} = [(f^{\sigma\circ\sigma^{-1}}, \sigma)]_{\approx} = [(f, \sigma)]_{\approx} = [(f, id)]_{\approx}^{\sigma} = \phi_{[n]}(f)^{\sigma}$$

and

$$\phi_{[n]}(f \diamond_i g) = [f \circ_i g, id]_{\approx} = [(f, id)]_{\approx} \circ_i [(g, id)]_{\approx} = \psi_{[n]}(f) \circ_i \psi_{[n]}(g),$$

which shows that the natural transformation $\phi : \mathcal{O} \to (\mathcal{O}_{ns})_s$, with components $\psi_{[n]}$, is indeed an isomorphism of operads $\mathcal{O}$ and $(\mathcal{O}_{ns})_s$.

The isomorphism of operads $\mathcal{O}$ and $(\mathcal{O}_s)_{ns}$. The bijection ψ_X between the sets $\mathcal{O}(X)$ and $(\mathcal{O}_s)_{ns}(X)$ is defined by

$$\psi_X : f \mapsto [(f^{\varphi_X^{-1}}, \varphi_X)]_{\approx},$$

where, assuming that $|X| = n$, $\varphi_X : X \to [n]$ is an arbitrary bijection. To see that ψ_X is well-defined, notice first that $f^{\varphi_X^{-1}} \in \mathcal{O}(n)$, i.e. that $[(f^{\varphi_X^{-1}}, \varphi_X)]_{\approx}$ is indeed an element of

$$\begin{array}{rcl}(\mathcal{O}_s)_{ns}(X) & = & \{[(g, \varphi_X)]_{\approx} \,|\, g \in \mathcal{O}_s(n) \text{ and } \varphi_X : X \to [n]\} \\ & = & \{[(g, \varphi_X)]_{\approx} \,|\, g \in \mathcal{O}(n) \text{ and } \varphi_X : X \to [n]\},\end{array}$$

and that, by (A.1.1), any other choice of φ_X would lead to the same equivalence class in the definition of ψ_X. The remaining of the (non-skeletal) operad structure transfers via ϕ_X as follows: for a bijection $\sigma : Y \to X$ we have

$$\begin{array}{rcl}\psi_X(f^{\sigma}) & = & [((f^{\sigma})^{\varphi_Y^{-1}}, \varphi_Y)]_{\approx} \\ & = & [(f^{\sigma\circ\varphi_Y^{-1}\circ\varphi_Y\circ\sigma^{-1}\circ\varphi_X^{-1}}, \varphi_X\circ\sigma)]_{\approx} \\ & = & [(f^{\varphi_X^{-1}}, \varphi_X\circ\sigma)]_{\approx} \\ & = & [(f^{\varphi_X^{-1}}, \varphi_X)]_{\approx}^{\sigma} \\ & = & \psi_X(f)^{\sigma},\end{array}$$

and the composition transfers as

$$\begin{array}{rcl}\psi_X(f) \circ_x \psi_X(g) & = & [(f^{\varphi_X^{-1}}, \varphi_X)]_{\approx} \circ_x [(g^{\varphi_Y^{-1}}, \varphi_Y)]_{\approx} \\ & = & [f^{\varphi_X^{-1}} \diamond_{\varphi_X(x)} g^{\varphi_Y^{-1}}, \varphi_Z]_{\approx} \\ & = & [(f^{\varphi_X^{-1}})^{\sigma_1} \circ_{\varphi_X(x)} (g^{\varphi_Y^{-1}})^{\sigma_2}, \varphi_Z]_{\approx} \\ & = & [((f \circ_x g)^{\varphi_Z^{-1}}, \varphi_Z)]_{\approx} \\ & = & \psi_X(f \circ_x g),\end{array}$$

where $Z = X\backslash\{x\} \cup Y$, $\varphi_Z : X\backslash\{x\} \cup Y \to [n+m-1]$ is defined as in (A.1.2), and $\sigma_1 : \{1, \dots, i\} \cup \{i+m, \dots, n+m-1\} \to [n]$ and $\sigma_2 : \{i, i+1, \dots, i+m-1\} \to [m]$ are defined as in (A.1.3). Notice that

$$\varphi_Z^{-1} = (\varphi_X \circ \sigma_1)|^{X\backslash\{x\}} \cup (\varphi_Y^{-1} \circ \sigma_2),$$

which establishes the equality

$$(f^{\varphi_X^{-1}})^{\sigma_1} \circ_{\varphi_X(x)} (g^{\varphi_Y^{-1}})^{\sigma_2} = (f \circ_x g)^{\varphi_Z^{-1}}$$

as an instance of the equivariance of $\mathcal{O}$. ∎

A.2 The good side of non-skeletality

We end this section with a comment on the advantages of the non-skeletal framework for cyclic operads. As the first benefit, we point out that, as opposed to the skeletal approach, the non-skeletal approach allows the entries-only presentation of non-symmetric cyclic operads (without the action of symmetric group, there is no way to formalise commutativity with numbered entries!). In turn, given that the entries-only definition of categorified cyclic operads is more compact than the exchangeable-output definition (compare Definition 4.1 and Definition 4.40), the non-skeletal approach is more economical in the categorified setting.

Also, non-skeletality turns out to be crucial for the rewriting involved in our proof of coherence in the presence of symmetries in Section 4.1. Namely, in the non-skeletal setting of (cyclic) operads, an action of the symmetric group can always be "pushed" from the composite of two operations to the operations themselves, by directing the equivariance law in the appropriate way. This was essential for the *first reduction* made in §4.1.3. For the skeletal setting of (cyclic) operads, this distribution of actions of the symmetric group doesn't work in general, as we illustrate in the example below.

Example A.2. Let $\mathcal{O} : \Sigma^{op} \to \mathbf{Set}$ be a (skeletal) operad. Let $f, g \in \mathcal{O}(2)$, and let $\sigma : [3] \to [3]$ be a permutation defined by $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(3) = 3$. Notice that there is a canonical embedding

$$\mathcal{O}(n) \ni h^{\tau} \mapsto [(h, \tau)]_{\approx} \in \mathcal{O}_{ns}(n)$$

and consider the term $(f \circ_2 g)^{\sigma}$. Clearly, it is not possible to distribute σ on f and g in $\mathcal{O}(3)$. However, with the above embedding, we get

$$\mathcal{O}(3) \quad \ni \quad (f \diamond_2 g)^{\sigma} \quad \mapsto \quad [(f \diamond_2 g, \sigma)]_{\approx} = [(f \diamond_2 g, id_{[3]})]_{\approx}^{\sigma} \quad \in \quad \mathcal{O}_{ns}(3).$$

In $\mathcal{O}_{ns}(3)$, the distribution of σ works as follows:

$$[(f \diamond_2 g, id_{[3]})]_{\approx}^{\sigma} = ([(f, id_{[2]})]_{\approx} \circ_2 [(g, \tau)]_{\approx})^{\sigma} = ([(f, id_{[2]})]_{\approx}^{\sigma_1} \circ_{2'} [(g, \tau)]_{\approx}^{\sigma_2},$$

where

- $\tau : \{2, 3\} \to \{1, 2\}$ is defined by $\tau(2) = 1$ and $\tau(3) = 2$,
- $\sigma_1 : \{2, 2'\} \to \{1, 2\}$ is defined by $\sigma_1(2) = 1$ and $\sigma_1(2') = 2$,
- $\sigma_2 : \{1, 3\} \to \{2, 3\}$ is defined by $\sigma_2(1) = 2$ and $\sigma_2(3) = 3$,
- the first equality holds by the definition of the composition operation $\circ_2$ in $\mathcal{O}_{ns}(n)$, and
- the second equality holds by the equivariance axiom `[EQ]` of $\mathcal{O}_{ns}$.

Therefore, denoting $\sigma_3 = \tau \circ \sigma_2$, we have $[(f \diamond_2 g, \sigma)]_{\approx} = [(f, \sigma_1)]_{\approx} \circ_{2'} [(g, \sigma_3)]_{\approx}$. □

Additionally, we are not sure whether orienting the equivariance in the opposite direction would work for the coherence proof. As a consequence, as we pointed out in §4.2.2, we prove skeletal coherence in the presence of symmetries by reducing it to the non-skeletal one.

Appendix B

Disjoint union vs. Union of disjoint sets

If one gives the priority to the non-skeletal operadic framework, one should be aware of subtleties concerning the choice between the disjoint union and the (ordinary) union of (already) disjoint sets as *the union* employed on the objects of **Bij** in the definition of an operad. In the litterature, one does not typically find definitions based on the latter kind of union, given that it does not exist for any two finite sets, and, therefore, involves bookkeeping the disjointness conditions, which is not the case with the coproduct. On the other hand, if one goes for coproducts, which is the practice in category theory, then one has to address the difficulties tied to non-associativity of disjoint union. In the usual implementation of disjoint union, given by the "preventive tagging" of sets, the sets

$$(X\backslash\{x\}+Y\backslash\{y\})+Z \qquad \text{and} \qquad X\backslash\{x\}+(Y\backslash\{y\}+Z)$$

are not equal, which, in turn, makes the sequential associativity law

$$(f\circ_x g)\circ_y h = f\circ_x (g\circ_y h)$$

ill-typed. This issue is the reason why we prefer to work with the ordinary union in the operadic context. Nevertheless, the subtlety of disjoint union is easily sorted out and the purpose of this appendix is to illustrate a proper definition of non-skeletal operads in the framework with coproducts.

The approach we take is inspired by the description of operads with *simultaneous* composition as monoids in the monoidal category of polynomial functors, given by Kock in [Koc11, Proposition 2.5.5]. The idea for recasting Definition 1.1 in the framework with disjoint union is to supplement the operations of an operad with an additional labeling of the inputs, as is done when evaluating polynomial functors. Essentially, with the definition that follows, we describe the monoidal structure corresponding to *partial* composition of polynomial functors, by spelling it out in the biased manner.

For arbitrary finite sets X and Y (for which, in particular, there are no disjointness assumptions), the composition structure of an operad $\mathcal{O}$, defined in the *polynomial style*, is given as follows. For $f\in\mathcal{O}(X)$, $x\in X$ and $g\in\mathcal{O}(Y)$, *the composition of f and g along x* is given by a triple

$$(U,k,\Phi)$$

where

- U is a finite set,
- $k\in\mathcal{O}(U)$, and
- $\Phi=[\varphi_f,\varphi_g]:X\backslash\{x\}+Y\to U$ is a bijection.

Denoting $U=U(f,x,g)$, $k=f\diamond_x g$, $\varphi_f=\varphi_f(f,x,g)$ and $\varphi_g=\varphi_g(f,x,g)$, as axioms we require the following equalities.

Firstly, we ask for the (adapted) associativity axioms:

`<A1>` $(f \diamond_x g) \diamond_{\varphi_g(f,x,g)(y)} h = f \diamond_x (g \diamond_y h)$, where $y \in Y$

`<A2>` $(f \diamond_x g) \diamond_{\varphi_f(f,x,g)(y)} h = (f \diamond_y h) \diamond_{\varphi_f(f,x,g)(x)} g$, where $y \in X$,

wherein, for both `<A1>` and `<A2>`, it is a priori required that, for some finite set U, the two sides of the respective equalities both belong to $\mathcal{O}(U)$. This is accomplished by imposing additional coherence conditions for the bijections figuring in the definitions of compositions involved in these two equalities. These conditions are easily derivable. For example, for `<A2>`, assuming that $h \in \mathcal{O}(Z)$, we ask that the following three equalities hold:

$$\begin{array}{rl} \varphi_{f\diamond_y h}(f \diamond_y h, \varphi_f(f,y,h), g)|_{\varphi_f(f,y,h)(X\backslash\{x,y\})} \circ \varphi_f(f,y,h)|_{X\backslash\{x,y\}} & = \\ \varphi_{f\diamond_x g}(f \diamond_x g, \varphi_f(f,x,g)(y), h)|_{\varphi_f(f,x,g)(X\backslash\{x,y\})} \circ \varphi_f(f,x,g)|_{X\backslash\{x,y\}} & , \end{array}$$

$$\varphi_{f\diamond_x g}(f \diamond_x g, \varphi_f(f,x,g)(y), h)|_{\varphi_g(f,x,g)(Y)} \circ \varphi_g(f,x,g) = \varphi_g(f \diamond_y h, \varphi_f(f,y,h)(x), g)$$

and

$$\varphi_{f\diamond_y h}(f \diamond_y h, \varphi_f(f,y,h), g)|_{\varphi_h(f,y,h)(Z)} \circ \varphi_h(f,y,h) = \varphi_h(f \diamond_x g, \varphi_f(f,x,g)(y), h).$$

Next, we require the (adapted) equivariance axiom: for the composition of f^{σ_1} and g^{σ_2} along $\sigma_1^{-1}(x)$ determined by the triple $(U, f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} g^{\sigma_2}, \Phi)$, where $\sigma_1 : X' \to X$ and $\sigma_2 : Y' \to Y$, and the composition of f and g along x determined by the triple $(V, f \diamond_x g, \Psi)$, we have

`<EQ>` $f^{\sigma_1} \diamond_{\sigma_1^{-1}(x)} g^{\sigma_2} = (f \diamond_x g)^{\sigma}$,

where the bijection $\sigma : U \to V$ is defined by $\sigma = \Psi \circ [\sigma_1|^{X\backslash\{x\}}, \sigma_2] \circ \Phi^{-1}$.

Finally, considering units, the axioms have the same form as `[U1]`, `[U2]` and `[UP]`. Similarly as in the case of associativity, these axioms a priori require that, for $f \in \mathcal{O}(X)$, the equalities $U(id_y, y, f) = X = U(f, x, id_x)$ hold. This is ensured by the following equalities: for `[U1]`, we additionally require that

$$\varphi_f(id_y, y, f) = id_X,$$

while, for `[U2]`, we ask that

$$\varphi_{id_x}(f, x, id_x) : X\backslash\{x\} + \{x\} \to X$$

is the canonical "forgetting-the-tags" bijection.

The characterisation of operadic composition we gave above is equivalent with the one used in Definition 1.1.

Indeed, starting from an operad $\mathcal{O}$, defined in the style of Definition 1.1, the composition structure of an operad in the polynomial style is obtained as follows. For arbitrary finite sets X and Y and an element $x \in X$, let

$$i_{X\backslash\{x\}} : X\backslash\{x\} \to X\backslash\{x\} + Y \qquad \text{and} \qquad i_Y : Y \to X\backslash\{x\} + Y$$

be the canonical injections of $X\backslash\{x\}$ and Y, respectively, into their disjoint union. Denote with Z_1 and Z_2 the images of $X\backslash\{x\}$ and Y under $i_{X\backslash\{x\}}$ and i_Y, respectively, and let

$$\sigma_1 : Z_1 \cup \{x\} \to X \qquad \text{and} \qquad \sigma_2 : Z_2 \to Y$$

be the bijections defined as

$$\sigma_1(1,u) = u, \;\; \sigma_1(x) = x \qquad \text{and} \qquad \sigma_2(2,v) = v.$$

Then, for operations $f \in \mathcal{O}(X)$ and $g \in \mathcal{O}(Y)$, we define the composition of f and g along x as the triple

$$(Z_1 \cup Z_2, f^{\sigma_1} \circ_x g^{\sigma_2}, [\varphi_f(f,x,g), \varphi_g(f,x,g)]),$$

where $[\varphi_f(f,x,g), \varphi_g(f,x,g)] = id_{X\backslash\{x\}+Y}$.

In the other direction, if $\mathcal{O}$ is an operad with composition structure defined in the polynomial style, and if for finite sets X and Y and an element $x \in X$ we have that $X\backslash\{x\} \cap Y = \emptyset$, let

$$\sigma_{X,x,Y} : X\backslash\{x\} \cup Y \to X\backslash\{x\} + Y$$

be the canonical "tagging" bijection. Then, for $f \in \mathcal{O}(X)$ and $g \in \mathcal{O}(Y)$, we define

$$f \circ_x g = (f \diamond_x g)^{[\varphi_f(f,x,g), \varphi_g(f,x,g)] \circ \sigma_{X,x,Y}}.$$

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